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The renormalized ϕ_4^4-trajectory by perturbation theory in a running coupling using partial differential equations

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Abstract

We compute the renormalized trajectory of ϕ_4^4-theory by perturbation theory in a running coupling. We use an exact infinitesimal renormalization group. The expansion is put into a form which is manifestly independent of the scale parameter.

1 Introduction

In Wilson's renormalization group [W71, WK74], renormalized theories come as renormalized trajectories of effective actions. The renormalization group leaves invariant a renormalized action up to a flow of renormalization parameters. This transformation law *defines* a renormalized theory. It opens a way to a formulate a renormalized theory without the detour to a limit procedure.

We study the ϕ^4-theory in four dimensional Euclidean space time. A dimension parameter D will be kept in the equations to display the dimension dependence of scale factors. We use a renormalization group transformation $\mathcal{R}_L$ which depends on a scale parameter L. The scale parameter is equal to the ratio of an ultraviolet and an infrared cutoff. We use an exact renormalization group differential equation for the L-dependence of effective actions. On the renormalized ϕ^4-trajectory the effective interaction depends on L through an L-dependent running coupling $g(L)$ only. A natural choice for g is the effective ϕ^4-coupling. On the renormalized trajectory, the normal ordered effective potential $\mathcal{V}(\phi, g)$ (see (53)) satisfies the renormalization group differential equation

$$\left\{\beta(g)\frac{\partial}{\partial g} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\right\}\mathcal{V}(\phi, g) = -\left\langle \mathcal{V}(\phi, g), \mathcal{V}(\phi, g)\right\rangle . \tag{1}$$

It is the main dynamical equation in our approach. Its most important property is *independence* of L. The renormalization group leaves invariant the renormalized trajectory up to a flow of g. This flow is encoded in a β-function through an ordinary differential equation

$$L\frac{\mathrm{d}}{\mathrm{d}L}g(L) = \beta(g(L)). \tag{2}$$

The operator $\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)$ generates a scale transformation. The right hand side of (1) is a bilinear renormalization group bracket $\langle A(\phi), B(\phi)\rangle$ defined in (55). It consists of contractions between $A(\phi)$ and $B(\phi)$, and is independent of g. The ϕ^4-trajectory is the solution to (1) with

$$\mathcal{V}(\phi, g) = g\frac{1}{4!}\int \mathrm{d}^D x \phi(x)^4 + g\frac{\zeta^{(1)}}{2}\int \mathrm{d}^D x \partial_\mu\phi(x)\partial_\mu\phi(x) + O(g^2). \tag{3}$$

It is unique in a space of *finite* solutions. To be precise, it is unique to all orders of perturbation theory in g. Solution is here meant as a formal power series in g with polynomial coefficients in ϕ. The criterion of finiteness is that the polynomials are given by smooth kernels on momentum space. Precise definitions are given in the bulk of this paper. The first order wave function term in (3) is peculiar to the four

dimensional case. The coupling $\zeta^{(1)}$ follows when (1) is expanded to second order in g.

The program of this paper is an iterative solution of (1) and of (2) in powers of g. In its course we do not encounter any divergencies. The result is a perturbation theory for the ϕ^4-trajectory which is finite to every order in g. This approach was proposed in [Wi96] using an iterated transformation with fixed L. In this paper we investigate the dependence upon smooth variation of L. It gives a refined formulation based on the differential equation (1). We restrict our attention to a one parametric family of renormalized potentials. It is distinguished by the property that no other vertices are included besides those generated dynamically from the first order ϕ^4-vertex. The outcome is then a one-dimensional curve, the ϕ^4-trajectory. We write it as a curve of effective potentials whose ultraviolet cutoff is rescaled to unity. To make contact with the physical world we would have to supply both a physical scale Λ and a value of g.

We included elementary details to make this paper self-contained. The experts in the field are begged pardon and asked to jump directly to what we call local perturbation theory. This paper is organized as follows. In section one we formulate the setup, make the connection with [Wi96], and give a derivation of the infinitesimal renormalization group. Two comments on the ultraviolet limit in the rescaled setup and the relation between Green's functions and the effective interaction are included. In section two we translate the renormalization group flow into the normal ordered representation. The second section closes with a bound on the bilinear renormalization group bracket. In section three we perform first an un-renormalized global perturbation expansion as a comparison, and thereafter the renormalized cutoff-free local perturbation expansion. Explicit second order calculations are included to demonstrate the methods. In the section four, the renormalization group equations are solved. We present a weak bound on the large momentum growth of the Euclidean Green's functions. It ensures that the iteration is indeed free of divergencies.

2 Renormalization Group

The setup of this paper is a momentum space renormalization group for a Euclidean scalar field ϕ.

2.1 Renormalization Group Transformation

Consider the following renormalization group transformation $\mathcal{R}_L$, depending on a scale parameter $L > 1$. Let $\mathcal{R}_L$ be composed of a Gaussian fluctuation integral, with

covariance Γ_L and mean ψ, and a dilatation $\mathcal{S}_L$ of ψ. Let the fluctuation covariance be defined by

$$\widetilde{\Gamma}_L(p) = \frac{1}{p^2}\left\{\widetilde{\chi}(p) - \widetilde{\chi}(Lp)\right\}, \tag{4}$$

where $\widetilde{\chi}(p)$ is a momentum space cutoff function. A convenient choice is the exponential cutoff

$$\widetilde{\chi}(p) = \exp\left(-p^2\right). \tag{5}$$

It will be used in the following. Other choices however work as well, for instance Pauli-Villar-regularization. The cutoff function's purpose is to suppress momenta outside a momentum slice $L^{-1} \leq |p| \leq 1$. Eq. (4) defines a parameter dependent positive operator Γ_L on the subspace of $L_2(\mathbb{R}^D)$ consisting of functions $f(x)$ with zero mode $\widetilde{f}(0) = 0$. Let $\mathrm{d}\mu_{\Gamma_L}(\zeta)$ be the associated Gaussian measure with mean zero on field space. Recall its basic property

$$\int \mathrm{d}\mu_{\Gamma_L}(\zeta) \exp\left\{(\zeta, f)\right\} = \exp\left\{\frac{1}{2}(f, \Gamma_L f)\right\}, \tag{6}$$

and consult for example Glimm and Jaffe [GJ87] for further information.

Let the fluctuation integral of a Boltzmann factor $Z(\phi) = \exp\{-V(\phi)\}$ be defined as the average with respect to (6), shifted by an external field ψ. A convenient notation for this average is

$$\langle Z \rangle_{\Gamma_L,\psi} = \int \mathrm{d}\mu_{\Gamma_L}(\zeta) Z(\psi + \zeta). \tag{7}$$

This fluctuation integral can be derived from a multi-scale decomposition of a free massless scalar field. Multi-scale decompositions are reviewed by Gallavotti in [G85]. See also the recent lectures [BG95] by Benfatto and Gallavotti, and references therein. The momentum slice $L^{-1} \leq |p| \leq 1$ can be thought to label a rescaled portion of momentum space degrees of freedom. This portion is integrated out in (7). The integration of another portion is prepared for by a dilatation $\mathcal{S}_L$ of ψ. Let the dilatation be given by

$$\mathcal{S}_L \psi(x) = L^{1-\frac{D}{2}} \psi\left(\frac{x}{L}\right). \tag{8}$$

The exponent $\sigma = 1 - D/2$ is the scaling dimension of a free massless field. Anomalous rescaling will not be considered below. Non-anomalous rescaling applies at least to weak perturbations of a free field.

The renormalization group transformation is the composition of (7) with (8). For the Boltzmann factor it reads

$$\mathcal{R}_L^{\mathrm{Bol}} Z(\psi) = \langle Z \rangle_{\Gamma_L, \mathcal{S}_L \psi} \,. \tag{9}$$

It is not difficult find renormalization group flows where the Boltzmann factor $Z(\phi)$ is a polynomial in the field. In scalar field theory the matter of interest are non-polynomial flows of the form $Z(\phi) = \exp\{-V(\phi)\}$, where the potential $V(\phi)$ is approximately local and polynomial. In the following it is approximately quartic. The renormalization group transformation for the potential reads

$$\mathcal{R}_L^{\mathrm{Pot}} V(\psi) = -\log \langle \exp(-V) \rangle_{\Gamma_L, \mathcal{S}\psi} \,. \tag{10}$$

The superscripts ^{Bol} and ^{Pot} will be dropped in the following by neglect of notation. The below analysis will be done entirely in terms of the potential. The matter of stability bounds on the Boltzmann factor will not be addressed. The method will be perturbation theory. It is valid in some vicinity of the trivial fixed point $V_*(\phi) = 0$. The renormalization group transformation for the Boltzmann factor is identical with the linearized transformation for the potential at the trivial fixed point. The linearized renormalization group is responsible for the leading dynamical behavior at weak coupling. A number of elementary properties of (10) are conveniently derived from (9), which is why the Boltzmann factor is introduced here at all.

We restrict our attention to even potentials with $V(-\phi) = V(\phi)$. Notice that field parity is preserved by (10). Potentials differing by a field independent constant will be identified. We could also impose a normalization condition, for instance $V(0) = 0$. To maintain normalization (10) then has to be supplemented with subtraction of $\mathcal{R}_L V(0)$. Technically this constant is proportional to the volume, infinite in infinite volume. Therefore, (10) requires an intermediate volume cutoff to make sense. We will wipe this technicality under the carpet and leave (10) as it stands. This setup is identical with that in [Wi96] up to the scale parameter L, which is here variable.

2.2 Semi-Group Property

The composition of two renormalization group transformations with scale L is equal to one renormalization group transformation with scale L^2. Moreover, the renormalization group transformation (10) satisfies

$$\mathcal{R}_{L_1} \mathcal{R}_{L_2} = \mathcal{R}_{L_1 L_2}, \quad L_1, L_2 > 1, \tag{11}$$

$$lim_{L \to 1^+} \mathcal{R}_L = \mathrm{i}d. \tag{12}$$

The renormalization group therefore defines a representation of the semi-group of dilatations of $\mathbb{R}^D$ with scale factors $L > 1$ on the space of effective interactions. The proof of this semi-group property is

$$\begin{aligned}\mathcal{R}_{L_1}\mathcal{R}_{L_2}Z(\psi) &= \int \mathrm{d}\mu_{\Gamma_{L_1}}(\zeta_1)\int \mathrm{d}\mu_{\Gamma_{L_2}}(\zeta_2)\; Z\left(\mathcal{S}_{L_2}(\mathcal{S}_{L_1}\psi+\zeta_1)+\zeta_2\right)\\ &= \int \mathrm{d}\mu_{\mathcal{S}_{L_2}\Gamma_{L_1}\mathcal{S}_{L_2}^T}(\zeta_1)\int \mathrm{d}\mu_{\Gamma_{L_2}}(\zeta_2)\; Z\left(\mathcal{S}_{L_2}\mathcal{S}_{L_1}\psi+\zeta_1+\zeta_2\right)\\ &= \int \mathrm{d}\mu_{\mathcal{S}_{L_2}\Gamma_{L_1}\mathcal{S}_{L_2}^T+\Gamma_{L_2}}(\zeta)\; Z\left(\mathcal{S}_{L_2}\mathcal{S}_{L_1}\psi+\zeta\right)\\ &= \mathcal{R}_{L_1L_2}Z(\psi).\end{aligned} \tag{13}$$

It uses a dilatation- and a convolution identity for Gaussian measures. Both follow from the functional Fourier transform (6). It also uses the elementary properties

$$\mathcal{S}_{L_2}\Gamma_{L_1}\mathcal{S}_{L_2}^T+\Gamma_{L_2}=\Gamma_{L_1L_2}, \tag{14}$$

$$\mathcal{S}_{L_1}\mathcal{S}_{L_2}=\mathcal{S}_{L_1L_2} \tag{15}$$

of (4) and (8). The property (12) follows from $\lim_{L\to 1^+}\widetilde{\Gamma}_L(p)=0^+$, turning (6) into a functional δ-measure.

Let us remark that $\widetilde{\Gamma}_L(p)$ becomes negative for $0 < L < 1$. Gaussian integration with negative covariance requires further regularity of functionals on field space. Non-invertibility of the renormalization group has been recently emphasized by Benfatto and Gallavotti in [BG95]. Effective theories tend to require less parameters because the number of effective degrees of freedom shrinks in the course of a renormalization group flow. Inverse renormalization group transformations can however be given a meaning on the renormalized trajectory. It is one dimensional from the beginning. Furthermore, an interaction on the renormalized trajectory is always the renormalization group image of another one. Restricted to the renormalized trajectory, the renormalization group defines a representation of the full dilatation group.

Due to the semi-group property the iteration of renormalization group transformations with fixed scale is identical with an increase of the scale in a single transformation. The renormalization group associates with an initial Boltzmann factor $Z(\phi)$ an orbit

$$Z(\phi,L)=\mathcal{R}_LZ(\phi), \tag{16}$$

parametrized by $L > 1$. Due to the semi-group property it interpolates the sequence

$$\mathcal{R}_L^nZ(\phi)=\mathcal{R}_{L^n}Z(\phi), \tag{17}$$

obtained by iteration of $\mathcal{R}_L$. This interpolation is the motive of the present investigation, in conjunction with the previous work in [Wi96] on the discrete case. The continuous point of view has the advantage to allow for infinitesimal renormalization group transformations, which can be expected close to the identity.

2.3 Infinitesimal Renormalization Group

The infinitesimal renormalization group was invented by Wilson [WK74]. Its importance as *exact* dynamical implementation of scale transformations is receiving increasing recognition. See also Wegner's review [We76].

The renormalization group orbit (16) satisfies the remarkable functional differential equation

$$\left\{ L\frac{\partial}{\partial L} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi} \right) \right\} Z(\phi, L) = \frac{1}{2} \left(\frac{\delta}{\delta\phi}, \mathrm{C}\frac{\delta}{\delta\phi} \right) Z(\phi, L). \tag{18}$$

Here $\mathcal{D}$ is the generator of dilatations of the field ϕ. In real space it is given by

$$\mathcal{D}\phi(x) = \mathcal{S}_{L^{-1}} \left(L\frac{\partial}{\partial L}\mathcal{S}_L \right) \phi(x) = \frac{\partial}{\partial L}\mathcal{S}_L\phi(x)\bigg|_{L=1} = \left\{ 1 - \frac{D}{2} - x\frac{\partial}{\partial x} \right\} \phi(x). \tag{19}$$

Furthermore, C is a rescaled scale derivative of the fluctuation covariance. Its explicit expression is

$$C = \mathcal{S}_{L^{-1}} \left(L\frac{\partial}{\partial L}\Gamma_L \right) \mathcal{S}^T_{L^{-1}}. \tag{20}$$

For a general cutoff function of the form $\widetilde{\chi}(p) = F(p^2)$, the covariance (20) is diagonal in momentum space with eigenvalues $\widetilde{C}(p) = -2F'(p^2)$. In the case of the exponential cutoff (20), it is $C = 2\chi$. In particular, it is independent of L and positive. The proof of (19) is

$$\begin{aligned} L\frac{\partial}{\partial L}Z(\psi, L) &= L\frac{\partial}{\partial L}\int \mathrm{d}\mu_{\Gamma_L}(\zeta)\; Z(\psi_L + \zeta) \\ &= \left\{ \left(L\frac{\partial}{\partial L}\psi_L, \frac{\delta}{\delta\psi_L} \right) + \frac{1}{2}\left(\frac{\delta}{\delta\psi_L}, \left(L\frac{\partial}{\partial L}\Gamma_L \right) \frac{\delta}{\delta\psi_L} \right) \right\} \int \mathrm{d}\mu_{\Gamma_L}(\zeta)\; Z(\psi_L + \zeta) \\ &= \left\{ \left(\mathcal{S}_{L^{-1}} \left(L\frac{\partial}{\partial L}\mathcal{S}_L \right) \psi, \frac{\delta}{\delta\psi} \right) + \frac{1}{2}\left(\frac{\delta}{\delta\psi}, \mathcal{S}_{L^{-1}} \left(L\frac{\partial}{\partial L}\Gamma_L \right) \mathcal{S}^T_{L^{-1}}\frac{\delta}{\delta\psi} \right) \right\} \\ &\quad Z(\psi, L). \end{aligned} \tag{21}$$

It uses the change of covariance formula for Gaussian measures and a dilatation identity for functional derivatives. Here ψ_L stands for $\mathcal{S}_L\psi$. The first operator on the right hand side of (21) performs an infinitesimal dilatation of the field ψ. Without this dilatation term we would have a functional heat equation. For the following analysis it is important to rescale the field and thus carry along the dilatation term. We should however concede that the question of ultraviolet and infrared limit can be

approached in a non-rescaled formalism as well. There the concept of a renormalized trajectory is hidden in scaling properties of a limit theory.

It is then straight forward to derive a functional differential equation for the potential $V(\phi, L) = -\log(Z(\phi, L))$. It is given by

$$\left\{L\frac{\partial}{\partial L} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) - \frac{1}{2}\left(\frac{\delta}{\delta\phi}, C\frac{\delta}{\delta\phi}\right)\right\} V(\phi, L) = \frac{-1}{2}\left(\frac{\delta}{\delta\phi}V(\phi, L), C\frac{\delta}{\delta\phi}V(\phi, L)\right). \tag{22}$$

Here we have collected the linear terms on the left hand side. Notice that the linearization of (22) at zero potential coincides with (19). Variations of this flow equation have proved to be useful both in the study of perturbation theory and in numerical studies of renormalization group flows. Polchinski [P84] for instance uses a flow equation without dilatation term in his beautiful proof of perturbative renormalizability.

2.4 Ultraviolet Limit

Eq. (4) suggests that we are always performing an infrared limit upon iteration of (10). In the rescaled formalism ultraviolet and infrared limit are closely related. We include this comment on rescaling to prevent confusion at this point. Consider a general massless covariance

$$\widetilde{v}_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}(p) = \frac{1}{p^2}\left\{\widetilde{\chi}\left(\frac{p}{\Lambda_{\mathrm{UV}}}\right) - \widetilde{\chi}\left(\frac{p}{\Lambda_{\mathrm{IR}}}\right)\right\}. \tag{23}$$

with two sided cutoffs. The ultraviolet limit refers to sending the upper cutoff Λ_{UV} to ∞ in

$$\bar{Z}^{eff}_{\Lambda_{\mathrm{IR}}}(\bar{\psi}) = \mathcal{R}_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}} \bar{Z}^{bare}_{\Lambda_{\mathrm{UV}}}(\bar{\psi}) = \int \mathrm{d}\mu_{v_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}}(\bar{\zeta})\ \bar{Z}^{bare}_{\Lambda_{\mathrm{UV}}}(\bar{\psi} + \bar{\zeta}), \tag{24}$$

keeping the lower cutoff Λ_{IR} fixed at a renormalization scale. Equivalently, the ratio $L = \Lambda_{\mathrm{UV}}/\Lambda_{\mathrm{IR}}$ is sent to infinity at fixed lower cutoff Λ_{IR}. Define a rescaled bare Boltzmann factor by

$$\bar{Z}^{bare}_{\Lambda_{\mathrm{UV}}}(\bar{\phi}) = Z^{bare}(\mathcal{S}_{\Lambda_{\mathrm{UV}}}\bar{\phi}) = Z^{bare}(\phi). \tag{25}$$

The bare Boltzmann factor is here written in *units* of the ultraviolet scale Λ_{UV}; to be precise in terms of a rescaled field (and rescaled couplings). Analogously write the effective Boltzmann factor in units of the infrared scale Λ_{IR},

$$\bar{Z}^{eff}_{\Lambda_{\mathrm{IR}}}(\bar{\psi}) = Z^{eff}(\mathcal{S}_{\Lambda_{\mathrm{IR}}}\bar{\psi}) = Z^{eff}(\psi). \tag{26}$$

Then the renormalization group transformation for the rescaled quantities is precisely of the form (9),

$$Z^{eff}(\phi) = \mathcal{R}_L Z^{bare}(\psi). \tag{27}$$

Thus keeping the infrared cutoff fixed, the ultraviolet limit is indeed equivalent with an infrared limit for the rescaled system. The proof of (27) is

$$\begin{aligned} Z^{eff}(\psi) &= \bar{Z}^{eff}(\mathcal{S}_{\Lambda_{\mathrm{IR}}{}^{-1}}\psi) \\ &= \int \mathrm{d}\mu_{v_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}}(\bar{\zeta})\ \bar{Z}^{bare}_{\Lambda_{\mathrm{UV}}}(\mathcal{S}_{\Lambda_{\mathrm{IR}}{}^{-1}}\psi + \bar{\zeta}) \\ &= \int \mathrm{d}\mu_{\mathcal{S}_{\Lambda_{\mathrm{UV}}{}^{-1}}\Gamma_L \mathcal{S}^T_{\Lambda_{\mathrm{UV}}{}^{-1}}}(\bar{\zeta})\ Z^{bare}\left(\mathcal{S}_{\Lambda_{\mathrm{UV}}}(\mathcal{S}_{\Lambda_{\mathrm{IR}}{}^{-1}}\psi + \bar{\zeta})\right) \\ &= \int \mathrm{d}\mu_{\Gamma_L}(\zeta)\ Z^{bare}(\mathcal{S}_L\psi + \zeta). \end{aligned} \tag{28}$$

In practice it is convenient to put $\Lambda_{\mathrm{IR}} = 1$ in physical units. Then the outcome of the rescaled renormalization group is already the desired effective potential. The argument can be summarized in the following diagram:

$$\begin{tikzcd} \bar{Z}^{bare}_{\Lambda_{\mathrm{UV}}}(\bar{\phi}) \arrow[r, equal, "\mathcal{S}_{\Lambda_{\mathrm{UV}}}\bar{\phi} = \phi"] \arrow[d, "\mathcal{R}_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}"'] & Z^{bare}(\phi) \arrow[d, "\mathcal{R}_L"] \\ \bar{Z}^{eff}_{\Lambda_{\mathrm{IR}}}(\bar{\psi}) \arrow[r, equal, "\mathcal{S}_{\Lambda_{\mathrm{IR}}}\bar{\psi} = \psi"'] & Z^{eff}(\psi) \end{tikzcd}$$

In the discrete renormalization group built upon iteration of (9), rescaling can be viewed as a stack of these diagrams on top of each other.

2.5 Green's Functions

The effective potential considered here is the generating function of free propagator amputated connected Euclidean Green's functions. Let us also include a brief mention of this property for the sake of completeness. Let $\bar{W}^{eff}_{\Lambda_{\mathrm{IR}}}(\bar{J})$ be the generating function of the connected Green's functions with free propagator $v_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}$ and vertices $\bar{V}^{bare}_{\Lambda_{\mathrm{UV}}}(\bar{\phi})$. Then

$$\bar{V}^{eff}_{\Lambda_{\mathrm{IR}}}(\bar{\psi}) = \bar{W}^{eff}_{\Lambda_{\mathrm{IR}}}(\bar{J}) + \frac{1}{2}\left(\bar{J}, v_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}\bar{J}\right), \quad \bar{\psi} = v_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}\bar{J}. \tag{29}$$

Eq. (29) follows from

$$\begin{aligned} \exp\left\{-\bar{V}^{eff}_{\Lambda_{\mathrm{IR}}}(\bar{\psi})\right\} &= \int \mathrm{d}\mu_{v_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}}(\bar{\zeta})\ \exp\left\{-\bar{V}^{bare}_{\Lambda_{\mathrm{UV}}}(\bar{\psi}+\bar{\zeta})\right\} \\ = \exp\left\{\frac{-1}{2}\left(\bar{\psi}, v^{-1}_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}\bar{\psi}\right)\right\} &\int \mathrm{d}\mu_{v_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}}(\bar{\zeta})\ \exp\left\{-\bar{V}^{bare}_{\Lambda_{\mathrm{UV}}}(\bar{\zeta}) + \left(\bar{\zeta}, v^{-1}_{\Lambda_{\mathrm{IR}},\Lambda_{\mathrm{UV}}}\bar{\psi}\right)\right\}, \end{aligned} \tag{30}$$

which uses a shift identity for Gaussian measures. The free propagator amputated connected Green's functions are given by the kernels of the development

$$\bar{V}^{eff}_{\Lambda_{\rm IR}}(\bar{\psi}) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \mathrm{d}^D\bar{x}_1 \ldots \mathrm{d}^D\bar{x}_{2n}\ \bar{\psi}(\bar{x}) \ldots \bar{\psi}(\bar{x}_{2n})\ \bar{V}^{eff}_{\Lambda_{\rm IR},2n}(\bar{x}_1, \ldots, \bar{x}_{2n}), \tag{31}$$

and the non-amputated connected ones by

$$\bar{W}^{eff}_{\Lambda_{\rm IR}}(\bar{J}) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \mathrm{d}^D\bar{x}_1 \ldots \mathrm{d}^D\bar{x}_{2n}\ \bar{J}(\bar{x}) \ldots \bar{J}(\bar{x}_{2n})\ \bar{W}^{eff}_{\Lambda_{\rm IR},2n}(\bar{x}_1, \ldots, \bar{x}_{2n}). \tag{32}$$

The non-amputated connected Green's functions are reconstructed from the amputated ones through

$$\begin{gathered}\bar{W}^{eff}_{\Lambda_{\rm IR},2}(\bar{x}_1, \bar{x}_2) = -v_{\Lambda_{\rm IR},\Lambda_{\rm UV}}(\bar{x}_1 - \bar{x}_2) + \\ \int \mathrm{d}^D\bar{y}_1\, \mathrm{d}^D\bar{y}_2\ v_{\Lambda_{\rm IR},\Lambda_{\rm UV}}(\bar{x}_1 - \bar{y}_1)\ v_{\Lambda_{\rm IR},\Lambda_{\rm UV}}(\bar{x}_2 - \bar{y}_2)\ \bar{V}^{eff}_{\Lambda_{\rm IR},2}(\bar{y}_1, \bar{y}_2),\end{gathered} \tag{33}$$

and

$$\begin{gathered}\bar{W}^{eff}_{\Lambda_{\rm IR},2n}(\bar{x}_1, \ldots, \bar{x}_{2n}) = \\ \int \mathrm{d}^D\bar{y}_1 \ldots \mathrm{d}^D\bar{y}_{2n}\ v_{\Lambda_{\rm IR},\Lambda_{\rm UV}}(\bar{x}_1 - \bar{y}_1) \ldots v_{\Lambda_{\rm IR},\Lambda_{\rm UV}}(\bar{x}_{2n} - \bar{y}_{2n})\ \bar{V}^{eff}_{\Lambda_{\rm IR},2}(\bar{y}_1, \ldots, \bar{y}_{2n}).\end{gathered} \tag{34}$$

The connection between the rescaled amputated Green's functions and the non-rescaled amputated ones on the other hand is

$$\bar{V}^{ren}_{\Lambda_{\rm IR}}(\bar{\psi}) = V^{ren}(\psi), \quad \mathcal{S}_{\Lambda_{\rm IR}}\bar{\psi} = \psi. \tag{35}$$

The non-rescaled amputated Green's functions are therefore explicitely given by

$$\bar{V}^{ren}_{\Lambda_{\rm IR},2n}(\bar{x}_1, \ldots, \bar{x}_{2n}) = {\Lambda_{\rm IR}}^{n(1+D/2)} V_{2n}(\Lambda_{\rm IR}\bar{x}_1, \ldots, \Lambda_{\rm IR}\bar{x}_{2n}) \tag{36}$$

in terms of the rescaled ones. In the ultraviolet limit the infrared scale $\Lambda_{\rm IR}$ is kept fixed and (36) amounts to a finite rescaling. But then the dictionary is complete. To obtain non-rescaled non-amputated connected Green's functions from the effective potential one has to undo rescaling using (36) and thereafter undo amputation using (33) and (34). It is clear that the infrared scale $\Lambda_{\rm IR}$ is here an extra piece of information which has to be supplied from the outside.

3 Normal Ordering

We choose to represent the potential in terms of normal ordered products. The payoff of normal ordering is that the linear part of the flow equations simplifies to a scale derivative and a dilatation term. The price to pay is a more involved bilinear term.

3.1 Normal Ordering Operator

We introduce another covariance v, which will serve as normal ordering covariance. Let v be given by

$$\widetilde{v}(p) = \frac{1}{p^2}\widetilde{\chi}(p), \tag{37}$$

a massless covariance with unit ultraviolet cutoff but without infrared cutoff. It satisfies

$$v = \mathcal{S}_{L^{-1}}(v - \Gamma_L)\mathcal{S}_{L^{-1}}^T. \tag{38}$$

As is shown for instance in [Wi96], this is the fixed point condition for a flow of normal ordering. In dimensions $D > 2$ the infrared singularity of (37) is integrable. This integrability is sufficient for the below purposes, in particular for an estimate on the bilinear renormalization group bracket in eq. (54). Associated with (37) is a normal ordering operator acting as

$$: \mathcal{Z}(\phi) :_v = \exp\left\{\frac{-1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\}\mathcal{Z}(\phi) \tag{39}$$

on polynomials, more generally power series, of the field. The commutator of the normal ordering operator (39) with the dilatation operator in the functional differential equation is computed to

$$\exp\left\{\frac{-1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\}\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\} =$$
$$\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) + \frac{1}{2}\left(\frac{\delta}{\delta\phi}, C\frac{\delta}{\delta\phi}\right). \tag{40}$$

The intention with the normal ordering covariance (37) was to obtain this identity. Differentiation of (37) with respect to the scale parameter L supplies us with

$$\mathcal{D}v + v\mathcal{D}^T = -\mathcal{S}_{L^{-1}}\left(L\frac{\partial}{\partial L}\Gamma_L\right)\mathcal{S}_{L^{-1}}^T = -C. \tag{41}$$

From it we conclude that the commutator of the functional Laplacian, built from the normal ordering covariance, with the dilatation operator is given by

$$\left[\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right), \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\right] = \left(\frac{\delta}{\delta\phi}, \left(\mathcal{D}v + v\mathcal{D}^T\right)\frac{\delta}{\delta\phi}\right) = -\left(\frac{\delta}{\delta\phi}, C\frac{\delta}{\delta\phi}\right). \tag{42}$$

But this is the infinitesimal version of (40). Eq. (40) then follows by integration.

Since the normal ordering covariance (37) is independent of L, the flow equation (19) is equivalent to

$$\exp\left\{\frac{-1}{2}\left(\frac{\delta}{\delta\phi},v\frac{\delta}{\delta\phi}\right)\right\}\left\{L\frac{\partial}{\partial L}-\left(\mathcal{D}\phi,\frac{\delta}{\delta\phi}\right)\right\}\exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi},v\frac{\delta}{\delta\phi}\right)\right\}Z(\phi,L)=0. \tag{43}$$

This equivalence suggests a normal ordered representation for the Boltzmann factor. We write it in the form

$$Z(\phi,L)=:\mathcal{Z}(\phi,L):_v=\exp\left\{\frac{-1}{2}\left(\frac{\delta}{\delta\phi},v\frac{\delta}{\delta\phi}\right)\right\}\mathcal{Z}(\phi,L). \tag{44}$$

We call $\mathcal{Z}(\phi,L)$ normal ordered Boltzmann factor. Strictly speaking it is the pre-image of a normal ordered Boltzmann factor by the normal ordering operator (39). Therefore it is not decorated with normal ordering colons. The normal ordered Boltzmann factor then satisfies the first order functional differential equation

$$\left\{L\frac{\partial}{\partial L}-\left(\mathcal{D}\phi,\frac{\delta}{\delta\phi}\right)\right\}\mathcal{Z}(\phi,L)=0. \tag{45}$$

A noticeable feature of the normal ordered representation is that the exact renormalization group equation (45) exactly performs an infinitesimal scale transformation. It is equivalent to

$$L\frac{d}{dL}\mathcal{Z}(\phi_L,L)=0, \tag{46}$$

with rescaled field $\phi_L=\mathcal{S}_L\phi$. In terms of the potential the theory is not as simple as this due to the non-linearity of (22). We remark that in perturbation theory we have in every order to deal with no more than polynomials in ϕ. Normal ordering thus requires no more than a finite number of extra contractions with the normal ordering covariance. They will be shown to be finite.

3.2 Homogeneous Solutions

The scaling fields of the trivial fixed point are polynomial solutions to (45). They are given by homogeneous kernels. Let us have a brief look at them because they will be used to parametrize the potential. A detailed discussion of scaling fields can be found for instance in Wegner's review [We76]. A polynomial

$$\mathcal{O}(\phi,L)=\frac{1}{n!}\int \mathrm{d}^Dx_1\ldots\mathrm{d}^Dx_n\;\phi(x_1)\ldots\phi(x_n)\;\mathcal{O}_n(x_1,\ldots,x_n,L) \tag{47}$$

in the field ϕ is a solution to the flow equation (45) iff the kernel satisfies

$$\left\{ L\frac{\partial}{\partial L} - \sum_{m=1}^{n} (\mathcal{D}^T)^{(m)} \right\} \mathcal{O}_n(x_1, \ldots, x_n, L) =$$

$$\left\{ L\frac{\partial}{\partial L} - n\left(1 + \frac{D}{2}\right) - \sum_{m=1}^{n} x_m \frac{\partial}{\partial x_m} \right\} \mathcal{O}_n(x_1, \ldots, x_n, L) = 0. \tag{48}$$

Eq. (48) is a homogeneous scaling equation. It is apparent that the renormalization group flow is a pure scale transformation in this formulation. The general solution of (48) is

$$\mathcal{O}_n(x_1, \ldots, x_n, L) = L^{n(1+D/2)} \mathcal{O}_n(Lx_1, \ldots, Lx_n). \tag{49}$$

A homogeneous kernel of degree κ, $\mathcal{O}_n(Lx_1, \ldots, Lx_n) = L^\kappa \mathcal{O}_n(x_1, \ldots, x_n)$, yields an eigenvector of the dilatation generator with eigenvalue $n(1 + D/2) + \kappa$. It is called a scaling field. The eigenvalue is called its real space scaling dimension. The kernels with negative degree of homogeneity have powerlike fall-off at large distances and singularities at small distances. Kernels with positive degree of homogeneity are non-local and discarded in field theory. A proper mathematical setting for field theoretic kernels is that of symmetric distributions on n copies of real space, given by Fourier integrals

$$\mathcal{O}_n(x_1, \ldots, x_n, L) =$$

$$\int \frac{\mathrm{d}^D p_1}{(2\pi)^D} \cdots \frac{\mathrm{d}^D p_n}{(2\pi)^D} \exp\left(i \sum_{m=1}^{n} p_m x_m \right) (2\pi)^D \delta^{(D)} \left(\sum_{m=1}^{n} p_m \right) \widetilde{\mathcal{O}}_n(p_1, \ldots, p_{n-1}, L). \tag{50}$$

We will restrict our attention to translation invariant kernels. The δ-function due to conservation of total momentum is then conveniently factorized from the momentum space kernels. In the sequel it is understood that the nth momentum of an n-point kernel is $p_n = -\sum_{m=1}^{n-1} p_m$. Furthermore, it is understood that the momentum space kernels are symmetric functions of the $n-1$ momenta. Eq. (50) is a solution to (45) iff

$$\left\{ L\frac{\partial}{\partial L} - \left(D + n\left(1 - \frac{D}{2}\right)\right) + \sum_{m=1}^{n-1} p_m \frac{\partial}{\partial p_m} \right\} \widetilde{\mathcal{O}}_n(p_1, \ldots, p_{n-1}, L) = 0. \tag{51}$$

The general solution to (51) is of course given by

$$\widetilde{\mathcal{O}}_n(p_1, \ldots, p_{n-1}, L) = L^{D+n(1-D/2)} \mathcal{O}\left(\frac{p_1}{L}, \ldots, \frac{p_{n-1}}{L} \right). \tag{52}$$

The power-counting of a homogeneous momentum space kernel of degree $\widetilde{\kappa}$ is therefore $D + n(1 - D/2) - \widetilde{\kappa}$. A momentum derivative reduces the power-counting of a kernel by one unit. The relevant parts of kernels in field theory come as sums of scaling fields with positive momentum space scaling dimension. They are extracted by Taylor expansion in momentum space. Here field theoretic momentum space kernels will be required to be symmetric, Euclidean invariant, and regular at zero momentum.

3.3 Normal Ordered Potential

Normal ordering reduces the flow of a Boltzmann factor to a pure scale transformation. This is not the case for the potential. Nevertheless it is useful to write potentials in normal ordered form at least for the case of weak coupling, where normal ordered products are eigenvectors of the linearized renormalization group. Let us define

$$V(\phi, L) =: \mathcal{V}(\phi, L) :_v = \exp\left\{ \frac{-1}{2} \left(\frac{\delta}{\delta\phi}, v \frac{\delta}{\delta\phi} \right) \right\} \mathcal{V}(\phi, L). \tag{53}$$

in likeness to (44). This normal ordered potential then obeys the following non-linear functional differential equation

$$\left\{ L \frac{\partial}{\partial L} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi} \right) \right\} \mathcal{V}(\phi, L) = - \left\langle \mathcal{V}(\phi, L), \mathcal{V}(\phi, L) \right\rangle. \tag{54}$$

It will be the main dynamical equation of this investigation. The non-linearity consists of a bilinear renormalization group bracket

$$\begin{aligned} &\langle \mathcal{V}(\phi, L), \mathcal{V}(\phi, L) \rangle = \\ &\frac{1}{2} \left(\frac{\delta}{\delta\phi^1}, C \frac{\delta}{\delta\phi^2} \right) \exp\left\{ \left(\frac{\delta}{\delta\phi^1}, v \frac{\delta}{\delta\phi^2} \right) \right\} \mathcal{V}(\phi^1, L) \mathcal{V}(\phi^2, L) \Big|_{\phi^1 = \phi^2 = \phi}. \end{aligned} \tag{55}$$

Here ϕ^1 and ϕ^2 denote two independent copies of ϕ. The bilinear term consists of contractions between two copies of the potential. Notice that every contraction contains one *hard* line C and any number of *soft* lines v. Eq. (54) has been used before in a proof of Symanzik's improvement program [Wi88].

As a byproduct we obtain a normal ordered functional differential equation for renormalization group fixed points. It is given by

$$\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi} \right) \mathcal{V}_*(\phi) = \left\langle \mathcal{V}_*(\phi), \mathcal{V}_*(\phi) \right\rangle. \tag{56}$$

Renormalization group fixed points are *global* solutions to (56). Non-perturbative tools for their investigation would be a major break through in this theory. Presently our toolbox contains only the ϵ-expansion and numerical recipes for truncated systems. An investigation of (56) along these lines will be presented elsewhere.

Eq. (56) has a trivial solution $\mathcal{V}_*(\phi) = 0$, the free massless field. In the following we will restrict our attention to weak perturbations of this trivial fixed point. It is then appropriate to use iterative methods to solve (54).

The proof of (54) is

$$
\begin{aligned}
&\left\{L\frac{\partial}{\partial L} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\right\} \mathcal{V}(\phi, L) = \\
&\quad \left\{L\frac{\partial}{\partial L} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\right\} \exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\} V(\phi, L) = \\
&\quad - \exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\} \frac{1}{2}\left(\frac{\delta}{\delta\phi}V(\phi, L), C\frac{\delta}{\delta\phi}V(\phi, L)\right) = \\
&\quad - \exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\} \frac{1}{2}\left(\frac{\delta}{\delta\phi}\exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\} \mathcal{V}(\phi, L),\right. \\
&\qquad \left. C\frac{\delta}{\delta\phi}\exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi}, v\frac{\delta}{\delta\phi}\right)\right\}\mathcal{V}(\phi, L)\right) = \\
&\quad \frac{-1}{2}\left(\frac{\delta}{\delta\phi^1}, C\frac{\delta}{\delta\phi^2}\right) \exp\left\{\frac{1}{2}\left(\frac{\delta}{\delta\phi^1} + \frac{\delta}{\delta\phi^2}, v\left(\frac{\delta}{\delta\phi^1} + \frac{\delta}{\delta\phi^2}\right)\right)\right\} \\
&\qquad \exp\left\{\frac{-1}{2}\left(\frac{\delta}{\delta\phi^1}, v\frac{\delta}{\delta\phi^1}\right)\right\} \exp\left\{\frac{-1}{2}\left(\frac{\delta}{\delta\phi^2}, v\frac{\delta}{\delta\phi^2}\right)\right\} \\
&\qquad \mathcal{V}(\phi^1, L)\mathcal{V}(\phi^2, L)\Big|_{\phi^1=\phi^2=\phi} = \\
&\quad \frac{-1}{2}\left(\frac{\delta}{\delta\phi^1}, C\frac{\delta}{\delta\phi^2}\right) \exp\left\{\left(\frac{\delta}{\delta\phi^1}, v\frac{\delta}{\delta\phi^2}\right)\right\} \mathcal{V}(\phi^1, L)\mathcal{V}(\phi^2, L)\Big|_{\phi^1=\phi^2=\phi}
\end{aligned} \tag{57}
$$

Undoing the field rescaling (54), becomes

$$
L\frac{\mathrm{d}}{\mathrm{d}L}\mathcal{V}(\phi_L, L) = - \left\langle \mathcal{V}(\phi_L, L), \mathcal{V}(\phi_L, L)\right\rangle . \tag{58}
$$

Since C is positive definite, the left hand side of (58) is always negative. It follows that the scaled potential decreases by value under a scale transformation. Notice that the difference between the bilinear term in the non-normal ordered formulation (22) and the normal ordered one (54) is a re-normal ordering operator. In perturbation theory it can be seen to create additional loops with normal ordering covariance. Moreover, normal ordering covariances appear only in contractions of vertices and never at external legs.

3.4 Bilinear Renormalization Group Bracket

In the non-linear formulation the bilinear term is responsible for inhomogeneous terms in scaling equations to be considered below. Let us write it explicitely for even monomials $\mathcal{O}_{2n}(\phi)$ in the field ϕ, given by

$$\mathcal{O}_{2n}(\phi) = \frac{1}{(2n)!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_{2n}\ \phi(x_1) \ldots \phi(x_{2n})\ \mathcal{O}_{2n}(x_1, \ldots, x_{2n}). \tag{59}$$

Recall that the bilinear term does not depend on the scale L. The bilinear operation on two monomials of the form (59) can be decomposed into

$$\langle \mathcal{O}_{2n}(\phi), \mathcal{O}_{2m}(\phi) \rangle = \sum_{l=|n-m|}^{n+m-1} N_{n,m,l} \left(\mathcal{O}_{2n} \star \mathcal{O}_{2m} \right)_{2l} (\phi), \tag{60}$$

and is itself a sum of monomials

$$\begin{gathered} \left(\mathcal{O}_{2n} \star \mathcal{O}_{2m} \right)_{2l} (\phi) = \\ \frac{1}{(2l)!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_{2l}\ \phi(x_1) \ldots \phi(x_{2l})\ \left(\mathcal{O}_{2n} \star \mathcal{O}_{2m} \right)_{2l} (x_1, \ldots, x_{2l}), \end{gathered} \tag{61}$$

whose kernels are given by a multiple convolutions

$$\begin{aligned} & \left(\mathcal{O}_{2n} \star \mathcal{O}_{2m} \right)_{2l} (x_1, \ldots, x_{2l}) = \\ & \quad \frac{1}{2(2l)!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_{2(n+m-l)}\ C(y_1 - y_{n+m-l+1}) \prod_{k=2}^{n+m-l} v(y_k - y_{n+m-l+k}) \\ & \quad \Big\{ \mathcal{O}_{2n}(x_1, \ldots, x_{n-m+l}, y_1, \ldots, y_{n+m-l}) \\ & \qquad \mathcal{O}_{2m}(x_{n-m+l+1}, \ldots, x_{2l}, y_{n+m-l+1}, \ldots, y_{2(n+m-l)}) + \\ & \qquad ((2l)! - 1) \text{ permutations} \Big\}. \end{aligned} \tag{62}$$

The kernels are understood to be symmetric under permutations of their entries. The multiple convolution involves one hard propagator C and $n+m-l-1$ soft propagators v, which is at the same time the number of loops. Furthermore, (60) involves a combinatorial factor

$$N_{n,m,l} = \frac{(2l)!}{(n+m-l-1)!(n-m+l)!(m-n+l)!}, \tag{63}$$

coming from the number of ways in which the contractions can be made. Eq. (62) can be interpreted as result of the fusion of two vertices. In the process of fusion links are created, consisting of propagators.

We present an elementary estimate on this fusion product to get an idea of what kind of analytical properties can be expected for the effective potentials. The estimate uses an $L_{\infty,\epsilon}$-norm in momentum space. The estimate works at this point for $\epsilon \geq 0$, not too large. Later it will be used for $\epsilon > 0$ only. Notice to begin with that

$$\|\widetilde{C}\|_{\infty,-2\epsilon},\ \|\widetilde{v}\|_{1,-2\epsilon} < \infty, \tag{64}$$

for the propagators with exponential cutoff.[1] At $\epsilon = 0$ we have for instance $\|\widetilde{C}\|_{\infty} = 2$ and $\|\widetilde{v}\|_{\infty} \leq 2\pi^{D/2}/(D-2)$. If the Fourier transformed kernels now satisfy the bounds

$$\|\widetilde{\mathcal{O}}_{2n}\|_{\infty,\epsilon},\ \|\widetilde{\mathcal{O}}_{2m}\|_{\infty,\epsilon} < \infty, \tag{65}$$

that is, are finite in the $L_{\infty,\epsilon}$-norm,[2] then it follows that all the terms (62) in the decomposition of the bilinear operation have finite $L_{\infty,\epsilon}$-norms in momentum space. They obey

$$\|(\mathcal{O}_{2n} \star \mathcal{O}_{2m})^{\sim}_{2l}\|_{\infty,\epsilon} \leq \frac{1}{2}\ \|\widetilde{C}\|_{\infty,-2\epsilon}\ \|\widetilde{v}\|^{n+m-l-1}_{1,-2\epsilon}\ \|\widetilde{\mathcal{O}}_{2n}\|_{\infty,\epsilon}\ \|\widetilde{\mathcal{O}}_{2m}\|_{\infty,\epsilon}. \tag{66}$$

Therefore, the renormalization group flow preserves the $L_{\infty,\epsilon}$-norm of momentum space kernels for finite scales. It will be shown that the $L_{\infty,\epsilon}$-norm is also preserved in the iterative solution of (1). The estimate immediately follows from the Fourier transform

$$\begin{aligned}
&(\mathcal{O}_{2n} \star \mathcal{O}_{2m})^{\sim}_{2l}\,(p_1, \ldots, p_{2l-1}) = \\
&\quad \frac{1}{2(2l)!} \int \frac{\mathrm{d}^D q_1}{(2\pi)^D} \cdots \frac{\mathrm{d}^D q_{n+m-l-1}}{(2\pi)^D}\ \widetilde{C}(q_{n+m-l}) \prod_{k=1}^{n+m-l-1} \widetilde{v}(q_i) \\
&\quad \Big\{ \widetilde{\mathcal{O}}_{2n}(p_1, \ldots, p_{n-m+l}, q_1, \ldots, q_{n+m-l-1}) \\
&\qquad \widetilde{\mathcal{O}}_{2m}(p_{n-m+l+1}, \ldots, p_{2l}, -q_1, \ldots, -q_{n+m-l-1}) + \\
&\qquad ((2l)! - 1) \text{ permutations} \Big\}.
\end{aligned} \tag{67}$$

[1]Here $\|\widetilde{C}\|_{\infty,-2\epsilon} = \sup_{p\in\mathbb{R}^D}\{|\widetilde{v}(p)|e^{2\epsilon|p|}\}$ and $\|\widetilde{v}\|_{1,-2\epsilon} = (2\pi)^{-D}\int \mathrm{d}^D p|\widetilde{v}(p)|e^{2\epsilon|p|}$ denote the $L_{\infty,-2\epsilon}$- and $L_{1,-2\epsilon}$-norms in momentum space.

[2]The $L_{\infty,\epsilon}$-norm for the momentum space kernels is defined as $\|\widetilde{\mathcal{O}}_{2n}\|_{\infty,\epsilon} = \sup_{(p_1,\ldots,p_{2n})\in\mathcal{P}_{2n}}\{|\widetilde{\mathcal{O}}(p_1,\ldots,p_{2n})|e^{-\epsilon(|p_1|+\cdots+|p_{2n}|)}\}$ with $\mathcal{P}_{2n} = \{(p_1,\ldots,p_{2n}) \in \mathbb{R}^D \times \cdots \times \mathbb{R}^D | p_1 + \cdots + p_{2n} = 0\}$ the hyperplane of total zero momentum.

of (62). The δ-functions from translation invariance have again been removed. The sums of momenta in the kernels are zero through

$$p_{2l} = -\sum_{m=1}^{2l-1} p_m, \quad q_{n+m-l} = -\sum_{k=1}^{n-m+l} p_k - \sum_{k=1}^{n+m-l-1} q_k. \tag{68}$$

The idea with the parameter ϵ is to use part of the exponential large momentum decay of the fluctuation and normal ordering propagators to compensate a possible large momentum growth of the kernels. In the initial value problem for (54) with L_∞-bounded inital data this might seem unnecessary. For instance a pure ϕ^4-vertex is constant and thus L_∞-bounded. The evolution preserves L_∞-boundedness for all *finite* scales L. However we cannot expect the solution to be L_∞-bounded uniformly in L. The limit $L \to \infty$ requires a separate treatment of zero momentum derivatives and Taylor remainders of the non-irrelevant kernels. The price to pay is a growth in momentum space. In the four dimensional case it is polynomial in powers and logarithms of momenta. With an exponential bound we are very far on the safe side.

4 Perturbation Theory

In perturbation theory the effective potential comes in form of a power series

$$\mathcal{V}(\phi, L, g) = \sum_{r=1}^{\infty} \frac{g^r}{r!} \, \mathcal{V}^{(r)}(\phi, L) \tag{69}$$

in a coupling parameter g. The effective potential is here assumed to be zero to zeroth order. The effective potential is thus expanded around the trivial fixed point. In the sequel (69) will be treated as a formal power series in g. The important question of non-summability of (69) will not be addressed.

4.1 Global Perturbation Theory

We speak of global perturbation theory when the expansion parameter is independent of the scale L. This expansion is not appropriate in the limit $L \to \infty$. Divergent terms appear and call for renormalization. We nevertheless develop the global expansion to some detail to see power counting at work. Inserting (69) into the bilinear bracket (55) and organizing the result again in powers of g we obtain

$$\langle \mathcal{V}(\phi, L, g), \mathcal{V}(\phi, L, g) \rangle = \sum_{r=2}^{\infty} \frac{g^r}{r!} \sum_{s=1}^{r-1} \binom{r}{s} \left\langle \mathcal{V}^{(s)}(\phi, L), \mathcal{V}^{(r-s)}(\phi, L) \right\rangle. \tag{70}$$

Let us introduce the abbreviation $\mathcal{K}^{(r)}(\phi, L)$ for the sum of brackets to order r on the right hand side of (70). The effective potential (69) is a solution to the

renormalization group equation (54) in the sense of a formal power series in g iff it satisfies

$$\left\{L\frac{\partial}{\partial L} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\right\}\mathcal{V}^{(r)}(\phi, L) = -\mathcal{K}^{(r)}(\phi, L) \tag{71}$$

to every order $r \geq 1$. Notice that $\mathcal{K}^{(r)}(\phi, L)$ depends on $\mathcal{V}^{(s)}(\phi, L)$ to lower orders $1 \leq s \leq r-1$ only. Supplied with initial data, (71) can be integrated and yields a recursion relation for the coefficients in (69).

4.1.1 First Order

To first order (71) is a homogeneos scaling equation

$$\left\{L\frac{\partial}{\partial L} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\right\}\mathcal{V}^{(1)}(\phi, L) = 0. \tag{72}$$

It tells that the evolution is a pure scale transformation to first order. Consider the case of ϕ^4-theory. There the first order is given by a ϕ^4-vertex

$$\mathcal{V}^{(1)}(\phi, L) = \lambda(L)\frac{1}{4!}\int \mathrm{d}^D x\ \phi(x)^4. \tag{73}$$

The reader is invited to add a mass and a wave function term if he wishes. This ϕ^4-vertex is a solution to the homogeneous scaling equation (72) provided that the coupling flows according to

$$\left\{\frac{\mathrm{d}}{\mathrm{d}L} - (4-D)\right\}\lambda(L) = 0. \tag{74}$$

The first order evolution is therefore $\lambda(L) = L^{4-D}\lambda(1)$. The conclusion is of course that the ϕ^4-vertex is relevant in dimensions $D < 4$, marginal in $D = 4$, and irrelevant in $D > 4$. In three dimensions the coupling flows proportional to the scale L. It diverges as $L \to \infty$. To first order one already sees that global perturbation theory is unsuitable to perform this limit.

4.1.2 Second Order

The inhomogeneous term due to the renormalization group bracket of two ϕ^4-vertices is computed to

$$\mathcal{K}^{(2)}(\phi, L) = \lambda(L)^2 \int \mathrm{d}^D x \mathrm{d}^D y \left\{\frac{1}{2}\phi(x)\phi(y)\ C(x-y)\ v(x-y)^2 + \right.$$

$$\left. \frac{1}{4!}\phi(x)^2\phi(y)^2\ 6C(x-y)\ v(x-y) + \frac{1}{6!}\phi(x)^3\phi(y)^3\ 20C(x-y)\right\}. \tag{75}$$

Two rather obvious remarks are in place here. First, the bracket is polynomial if all lower order vertices are polynomials. Second, the highest power of fields in the bracket is the sum of powers of the fused vertices minus two. It is instructive to enter the second order equation (71) with an ansatz containing precisely the interactions present in (75),

$$\mathcal{V}^{(2)}(\phi, L) = \lambda(L)^2 \int \mathrm{d}^D x \mathrm{d}^D y \Big\{ \frac{1}{2}\phi(x)\phi(y)\ \mathcal{V}_2^{(2)}(x-y, L) + \frac{1}{4!}\phi(x)^2\phi(y)^2\ \mathcal{V}_4^{(2)}(x-y, L) + \frac{1}{6!}\phi(x)^3\phi(y)^3\ \mathcal{V}_6^{(2)}(x-y, L)\Big\}. \tag{76}$$

It yields a solution to (71) iff the kernels in the ansatz obey the inhomogeneous differential equations

$$\left\{ L\frac{\partial}{\partial L} - \sigma_{2n}^{(2)} - x\frac{\partial}{\partial x} \right\} \mathcal{V}_{2n}^{(2)}(x, L) = -\mathcal{K}_{2n}^{(2)}(x) \tag{77}$$

with

$$\mathcal{K}_2^{(2)}(x) = C(x)v(x)^2, \quad \mathcal{K}_4^{(2)}(x) = 6C(x)v(x), \quad \mathcal{K}_6^{(2)}(x) = 20C(x) \tag{78}$$

and

$$\sigma_2^{(2)} = 3(D-2), \quad \sigma_4^{(2)} = 2(D-2), \quad \sigma_6^{(2)} = D-2. \tag{79}$$

Eq. (77) is easily integrated. Let us perform the integration in momentum space and thereby prepare the ground for the local perturbation expansion below. Fourier transformation turns (77) into

$$\left\{ L\frac{\partial}{\partial L} - \widetilde{\sigma}_{2n}^{(2)} + p\frac{\partial}{\partial p} \right\} \widetilde{\mathcal{V}}_{2n}^{(2)}(p, L) = -\widetilde{\mathcal{K}}_{2n}^{(2)}(p) \tag{80}$$

with[3]

$$\widetilde{\mathcal{K}}_2^{(2)}(p) = \widetilde{C} \star \widetilde{v} \star \widetilde{v}(p), \quad \widetilde{\mathcal{K}}_4^{(2)}(p) = 6\widetilde{C} \star \widetilde{v}(p), \quad \widetilde{\mathcal{K}}_6^{(2)}(p) = 20\widetilde{C}(p), \tag{81}$$

and

$$\widetilde{\sigma}_2^{(2)} = 2D-6, \quad \widetilde{\sigma}_4^{(2)} = D-4, \quad \widetilde{\sigma}_6^{(2)} = -2. \tag{82}$$

[3]The convolution of two momentum space functions is here defined as $\widetilde{F} \star \widetilde{G}(p) = (2\pi)^{-D} \int \mathrm{d}^D q \widetilde{F}(p-q)\, \widetilde{G}(q)$.

Notice that our second order ansatz has a prefactor $\lambda(L)^2$. Therefore the scaling dimensions here are those of the corresponding monomials relative to the that of two ϕ^4-vertices. Another way to write the momentum space equation (80) is

$$L^{\widetilde{\sigma}_{2n}^{(2)}} L \frac{\mathrm{d}}{\mathrm{d}L} \left\{ L^{-\widetilde{\sigma}_{2n}^{(2)}} \widetilde{\mathcal{V}}_{2n}^{(2)}(Lp, L) \right\} = -\widetilde{\mathcal{K}}_{2n}^{(2)}(Lp). \tag{83}$$

In this form it is immediately integrated to

$$\widetilde{\mathcal{V}}_{2n}^{(2)}(p, L) = L^{\widetilde{\sigma}_{2n}^{(2)}} \widetilde{\mathcal{V}}_{2n}^{(2)}\left(\frac{p}{L}, 1\right) - \int_{L^{-1}}^{1} \frac{\mathrm{d}t}{t} t^{-\widetilde{\sigma}_{2n}^{(2)}} \widetilde{\mathcal{K}}_{2n}^{(2)}(tp). \tag{84}$$

From this expression we can learn that the evolution tends to forget the irrelevant part of the initial potential as $L >> 1$, whereas the relevant part of the initial potential is enhanced. We will not renormalize the second order flow at this instant. But let us remark that due to the damping of irrelevant initial data an ultraviolet limit depends only on non-irrelevant interactions in the bare potential. Furthermore we see that the the integral in (84) contains divergent terms in the non-irrelevant case as $L \to \infty$. Both powerlike and logarithmic singularities appear.

A general solution of the second order equation can be composed of this particular solution and any solution of the homogeneous scaling equation

$$\left\{ L\frac{\partial}{\partial L} - \widetilde{\sigma}_{2n}^{(2)} + p\frac{\partial}{\partial p} \right\} \widetilde{\mathcal{V}}_{2n}^{(2)}(p, L) = 0. \tag{85}$$

In the sequel we will adopt the following point of view regarding this freedom. The first order interaction enforces recursively through the bilinear bracket a certain set of vertices at higher orders. The homogeneous equation (85) allows us to introduce further vertices by hand into the iteration at higher orders, which are not present in a minimal scheme. We will restrict our attention to the solution to the renormalization group equation with a *minimal* set of vertices. This solution is determined by the first order potential. We can think of the first order as the germ of the theory. Introduction of other vertices at higher orders might however offer an interesting way to mix models.

4.1.3 Higher Orders

If the first order is polynomial, so are all higher orders in the minimal scheme. In the case of ϕ^4-theory the general form of higher order vertices is

$$\mathcal{V}^{(r)}(\phi, L) = \lambda(L)^r \sum_{n=1}^{r+1} \frac{1}{(2n)!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_{2n}\, \phi(x_1) \ldots \phi(x_{2n})\, \mathcal{V}_{2n}^{(r)}(x_1, \ldots, x_{2n}, L). \tag{86}$$

This general form iterates to every order of perturbation theory. Other interactions could however be introduced by hand. The highest connected vertex built from r first order ϕ^4-vertices has $2(r+1)$ fields. The renormalization group equation for the kernels in (86) reads

$$\left\{ L\frac{\partial}{\partial L} - \sigma_{2n}^{(r)} - \sum_{l=1}^{2n} x_l \frac{\partial}{\partial x_l} \right\} \mathcal{V}_{2n}^{(r)}(x_1, \dots, x_{2n}, L) = -\mathcal{K}_{2n}^{(r)}(x_1, \dots, x_{2n}) \tag{87}$$

with real space scaling dimensions $\sigma_{2n}^{(r)} = m(2+D) - r(4-D)$. The equivalent equation in momentum space is

$$\left\{ L\frac{\partial}{\partial L} - \widetilde{\sigma}_{2n}^{(r)} + \sum_{l=1}^{2n-1} p_l \frac{\partial}{\partial p_l} \right\} \widetilde{\mathcal{V}}_{2n}^{(r)}(p_1, \dots, p_{2n-1}, L) = -\widetilde{\mathcal{K}}_{2n}^{(r)}(p_1, \dots, p_{2n-1}) \tag{88}$$

with momentum space scaling dimension $\widetilde{\sigma}_{2n}^{(r)} = D + n(2-D) - r(4-D)$. A compact way of writing (88) then is

$$L^{\widetilde{\sigma}_{2n}^{(r)}} L\frac{\mathrm{d}}{\mathrm{d}L} \left\{ L^{-\widetilde{\sigma}_{2n}^{(r)}} \widetilde{\mathcal{V}}_{2n}^{(r)}(Lp_1, \dots, Lp_{2n-1}, L) \right\} = -\widetilde{\mathcal{K}}_{2n}^{(r)}(Lp_1, \dots, Lp_{2n-1}), \tag{89}$$

which is then integrated to

$$\begin{aligned} &\widetilde{\mathcal{V}}_{2n}^{(r)}(p_1, \dots, p_{2n-1}, L) = \\ &L^{\widetilde{\sigma}_{2n}^{(r)}} \widetilde{\mathcal{V}}_{2n}^{(r)} \left(\frac{p_1}{L}, \dots, \frac{p_{2n-1}}{L}, 1 \right) + \int_{L^{-1}}^{1} \frac{\mathrm{d}t}{t} t^{-\widetilde{\sigma}_{2n}^{(r)}} \widetilde{\mathcal{K}}_{2n}^{(r)}(tp_1, \dots, tp_{2n-1}). \end{aligned} \tag{90}$$

Thereby we have put perturbation theory for the renormalization group evolution into the form of a recursion relation. A recursion step consists of all contractions of previous vertices with one another to a given total order plus integration of (90).

As it stands this perturbation expansion develops singular coefficients when the scale L is taken to infinity. These singularities can be removed by renormalization. An elegant way to renormalize the series goes as follows. In the recursion formula (90) the integral is performed from the ultraviolet end to the infrared end of the theory. Polchinski [P84] integrates the non-irrelevant degrees of freedom precisely the other way using a mixed boundary value problem. The relevant data is there prescribed on a lower scale than the irrelevant data. We mention that this approach can be generalized to a version of Symanzik's improvement program [Wi88]. This renormalization technology is well developed by now.

In this paper we choose another route. We will search (and find) a way to formulate the theory in terms of quantities which are *independent* of the scale L. These quantities are auto-renormalized. They are identical with their scaling limits. The idea is switch to another form of perturbation theory.

4.2 Local Perturbation Theory

We speak of local perturbation theory when the expansion parameter g is taken to depend on the scale L. A particular form of local perturbation theory is the running coupling expansion for renormalized trajectories proposed in [Wi96]. There the ambition is to find renormalization group flows whose scale dependence comes exclusively in form of a running coupling $g(L)$. The expansion coefficients themselves are scale independent. We will develop such an expansion for the ϕ^4-trajectory in the infinitesimal renormalization group setup. A forerunner with discrete renormalization group transformation is found in [Wi96]. We intend to solve the flow equation (54) in terms of a power series

$$\mathcal{V}(\phi, g(L)) = \sum_{r=1}^{\infty} \frac{g(L)^r}{r!} \mathcal{V}^{(r)}(\phi). \tag{91}$$

The interaction $\mathcal{V}^{(r)}(\phi)$ is assumed to be independent of L. We choose the ϕ^4-coupling as expansion parameter. It is defined through the condition

$$\widetilde{\mathcal{V}}_4(0,0,0,g(L)) = g(L) \tag{92}$$

on the four point kernel at zero momentum. To be precise, we impose the perturbative constraint that (92) be zero for all orders larger than one. The ϕ^4-coupling is by no means the only possible choice in this approach. It is however a natural candidate when dealing with ϕ^4-theory. It will be used in the following. In order to solve (54) in terms of (91) we also require another power series

$$L\frac{\mathrm{d}}{\mathrm{d}L} g(L) = \beta\left(g(L)\right) = \sum_{r=1}^{\infty} \frac{g(L)^r}{r!} \beta^{(r)}. \tag{93}$$

The outcome of this analysis is a curve in the space of effective interactions parametrized by g. The β-function (93) says how the renormalization group acts on interactions on this curve. The coefficients $\beta^{(r)}$ form a second set of unknowns, besides those in $\mathcal{V}^{(r)}(\phi)$. They have to come out of the theory. To be more precise, the β-function substitutes those degrees of freedom which are removed by (92). The intent with (91) is a renormalization group orbit which is *not* parametrized by L but rather by g. As we will see such a curve is indeed determined to all orders of perturbation theory once we supply an appropriate first order interaction. In the case of ϕ^4-theory the appropriate first order interaction is

$$\mathcal{V}^{(1)}(\phi) = \frac{1}{4!} \int \mathrm{d}^D x \; \phi(x)^4, \tag{94}$$

a ϕ^4-vertex. The corresponding renormalization group orbit is called the ϕ^4-trajectory. It is the object of principle interest in massless ϕ^4-theory. Potentials on the ϕ^4-trajectory are said to scale. The first order interaction (94) turns out to require a

slight modification in four dimensions. We need to add also a wave function term. We will do this below.

4.2.1 Scaling Equations

We insert the expansion (91) into the renormalization group equation (54) and deduce a system of equations for the unknowns therefrom. This system is called the set of scaling equations. We have

$$L\frac{\partial}{\partial L}\mathcal{V}(\phi,g(L)) = \beta\left(g(L)\right)\frac{\partial}{\partial g(L)}\mathcal{V}(\phi,g(L)) = \sum_{r=1}^{\infty}\frac{g(L)^r}{r!}\sum_{s=1}^{r}\binom{r}{s}\beta^{(s)}\mathcal{V}^{(r-s+1)}(\phi). \tag{95}$$

Therefore, our local perturbation expansion is a solution to (54) in the sense of a formal power series iff the unknowns obey

$$\sum_{s=1}^{r}\binom{r}{s}\beta^{(s)}\mathcal{V}^{(r-s+1)}(\phi) - \left(\mathcal{D}\phi,\frac{\delta}{\delta\phi}\right)\mathcal{V}^{(r)}(\phi) = -\sum_{s=1}^{r-1}\binom{r}{s}\left\langle V^{(s)}(\phi),\mathcal{V}^{(r-s)}(\phi)\right\rangle \tag{96}$$

to all orders $r \geq 1$. The most remarkable property of (96) is again independence of L. Another way to think of (96) is that we are looking for a fixed point of the transformation composed of a renormalization group step and a transformation of the four point coupling. We organize (96) into a recursion relation.

4.2.2 First Order

Scaling requires of $\mathcal{V}^{(1)}(\phi)$ to be a scaling field of the trivial fixed point. To first order (96) reads

$$\left\{\beta^{(1)} - \left(\mathcal{D}\phi,\frac{\delta}{\delta\phi}\right)\right\}\mathcal{V}^{(1)}(\phi) = 0. \tag{97}$$

It requires of $\mathcal{V}^{(1)}(\phi)$ to be an eigenvector of the generator of dilatations, and of $\beta^{(1)}$ to be its eigenvalue. The ϕ^4-vertex (94) is indeed an eigenvector. Its eigenvalue has the familiar value

$$\beta^{(1)} = 4 - D. \tag{98}$$

Here three remarks are in place. First, if we want to add further terms to the first order (94) then (97) requires of all other terms to be eigenvectors with this same eigenvalue (98). In four dimensions the eigenvalue (98) is zero. There one has indeed another marginal eigenvector, the wave function term. Second, we could

have considered any other scaling field of the trivial fixed point as our first order starting point, for instance a ϕ^6-vertex in three dimensions. All of the scaling fields generate interesting trajectories, for which this theory equally well applies. Third, the eigenvalue does not need to be larger or equal to zero. We can as well start with an irrelevant scaling field. In fact the ϕ^4-vertex *is* marginally irrelvant in four dimensions.

4.2.3 Second Order

The first order equation (97) is special in that it is homogeneous. All higher order scaling equations are inhomogeneous. The second order equation is given by

$$\left\{2\beta^{(1)} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right)\right\} \mathcal{V}^{(2)}(\phi) = -\beta^{(2)}\mathcal{V}^{(1)}(\phi) - 2\left\langle \mathcal{V}^{(1)}, \mathcal{V}^{(1)} \right\rangle. \tag{99}$$

Notice that unlike in the global expansion there is as well an inhomogeneous term due to the flow of the coupling parameter. Minus the right hand side of (99) is explicitely computed to

$$\mathcal{K}^{(2)}(\phi) = \frac{\beta^{(2)}}{4!}\int \mathrm{d}^D x\phi(x)^4 + \int \mathrm{d}^D x \mathrm{d}^D y \Big\{\frac{1}{2}\phi(x)\phi(y)C(x-y)v(x-y)^2 + \frac{1}{4!}\phi(x)^2\phi(y)^2 6C(x-y)v(x-y) + \frac{1}{6!}\phi(x)^3\phi(y)^3 20C(x-y)\Big\}. \tag{100}$$

The general solution of (99) consists of special solution plus any solution of the homogeneous equation. In the global expansion special solutions were singled out by boundary conditions. Here we will do something different. We will only admit solutions which are given by finite kernels in momentum space. Furthermore, we will restrict our attention to local interactions in the minimal scheme. It then turns out that there exists only one such solution. This special solution contains precisely those vertices, which are enforced by the first order interaction. Therefore, we make the following ansatz

$$\mathcal{V}^{(2)}(\phi) = \sum_{n=1}^{3} \frac{1}{(2n)!}\int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_{2n}\ \phi(x_1)\ldots\phi(x_{2n})\ \mathcal{V}^{(2)}_{2n}(x_1,\ldots,x_{2n}). \tag{101}$$

From (100) it is clear that we can only succeed in terms of distributional kernels. We therefore Fourier transform (99) and thereby obtain

$$\left\{2\beta^{(1)} - (D - n(D-2)) + \sum_{m=1}^{2n-1} p_m \frac{\partial}{\partial p_m}\right\} \widetilde{\mathcal{V}}^{(2)}_{2n}(p_1,\ldots,p_{2n-1}) = -\widetilde{\mathcal{K}}^{(2)}_{2n}(p_1,\ldots,p_{2n-1}) \tag{102}$$

with inhomogeneous terms given by

$$\widetilde{\mathcal{K}}_2^{(2)}(p) = \widetilde{C} \star \widetilde{v} \star \widetilde{v}(p), \tag{103}$$

$$\widetilde{\mathcal{K}}_4^{(2)}(p_1, p_2, p_3) = \beta^{(2)} + 6\widetilde{C} \star \widetilde{v}(p_1 + p_2), \tag{104}$$

$$\widetilde{\mathcal{K}}_6^{(2)}(p_1, \dots, p_5) = 20\widetilde{C}(p_1 + p_2 + p_3). \tag{105}$$

The right hand sides are here understood to be symmetrized in the momenta. We do not write this symmetrization explicitely in order to simplify the notation. Here we recognize again the the second order scaling dimension $\widetilde{\sigma}_{2n}^{(2)} = D - n(D-2) - 2\beta^{(1)}$, which was introduced above. The general method to solve the differential equation (102) is explained below. We immediately apply it to this second order equation. Consider first the six point interaction. Since $\widetilde{\sigma}_6^{(2)} = -2$, it is irrelevant (in any dimension), we can immediately integrate the six point interaction to[4]

$$\widetilde{\mathcal{V}}_6^{(2)}(p_1, \dots, p_5) = -\int_0^1 \frac{\mathrm{d}t}{t} t^2 \widetilde{\mathcal{K}}_6^{(2)}(tp_1, \dots, tp_5). \tag{106}$$

Notice the similarity with the analogous expression in the global expansion. Formally it can be obtained by taking L to infinity in (90). We then treat the four point interaction. It is computed in two parts. The first part concerns its value at zero momentum. By definition of our expansion parameter we have to satisfy

$$0 = \beta^{(2)} + 6\widetilde{C} \star \widetilde{v}(0). \tag{107}$$

Using the ϕ^4-coupling to organize the expansion, we imposed that $\widetilde{\mathcal{V}}_4^{(2)}(0,0,0)$, and all higher orders of the four point kernel at zero momentum, be zero. Eq. (107) determines the value of $\beta^{(2)}$. In four dimensions its value is computed to

$$\beta^{(2)} = \frac{-6}{(4\pi)^2}. \tag{108}$$

As a consequence the ϕ^4-vertex is indeed marginally irrelevant in four dimensions. Thereby the renormalization group flow on the ϕ^4-trajectory is not asymptotically free in the ultraviolet. Nevertheless the ϕ^4-trajectory is a well defined object at weak coupling. The irrelevant remainder of the four point vertex is then integrated to

$$\widetilde{\mathcal{V}}_4^{(2)}(p_1, p_2, p_3) = -6\int_0^1 \frac{\mathrm{d}t}{t} t^{4-D} \left\{ \widetilde{C} \star \widetilde{v}(tp_1 + tp_2) - \widetilde{C} \star \widetilde{v}(0) \right\}. \tag{109}$$

[4] It is instructive to perform this integral. The result is $\widetilde{\mathcal{V}}_6^{(2)}(p_1, \dots, p_5) = 10(p_1 + p_2 + p_3)^{-2} \left\{ \exp\left(-(p_1 + p_2 + p_3)^2\right) - 1 \right\}$. This expression is regular at zero momentum and a bounded function of the momenta. It is of the form of a cutoff propagator.

Here it becomes transparent how the flow of the coupling parameter saves us from a logarithmic singularity in four dimensions. Notice also that the subtraction of the zero momentum piece is unnecessary in three dimensions. There this vertex is already irrelevant. The two point kernel finally requires most attention and poses even an obstacle at this stage. Due to Euclidean invariance we have

$$\widetilde{\mathcal{V}}_2^{(2)}(p) = A(p^2), \quad \widetilde{\mathcal{K}}_2^{(2)}(p) = 2B(p^2). \tag{110}$$

It is convenient to trade p^2 for a new variable u. The scaling dimension of the two point kernel is $\widetilde{\sigma}_2^{(2)} = 2D - 6$, and is two in four dimensions. The scaling equation for the two point kernel then becomes

$$\left\{ u \frac{\mathrm{d}}{\mathrm{d}u} - 1 \right\} A(u) = -B(u). \tag{111}$$

Its solution requires a second order Taylor expansion with remainder term for the function

$$A(u) = A(0) + u\, A'(0) + \frac{u^2}{2} \int_0^1 \mathrm{d}t (1-t) A''(tu). \tag{112}$$

The zero momentum value is directly determined. Its value is

$$A(0) = B(0) = \frac{1}{2} \widetilde{C} \star \widetilde{v} \star \widetilde{v}(0) = \frac{1}{(4\pi)^4} \left(2 \log(2) - \log(3) \right), \quad D = 4. \tag{113}$$

It can be interpreted as a second order mass parameter. The Taylor remainder in (112) is computed from the second derivative of (111). We have

$$\left\{ u \frac{\mathrm{d}}{\mathrm{d}u} + 1 \right\} A''(u) = -B''(u). \tag{114}$$

Two u-derivatives have changed the powercounting by four units. As a consequence the second u-derivative is irrelevant and is therefore integrated to

$$A''(u) = - \int_0^1 \mathrm{d}t\, B''(tu) \tag{115}$$

in complete analogy to the case of the six point kernel.[5] The grain of salt is the first u-derivative, a wave function term. Its scaling equation is

$$u \frac{\mathrm{d}}{\mathrm{d}u} A'(u) = -B'(u). \tag{116}$$

[5] A parameter representation for the inhomogeneous mass term is $\widetilde{\mathcal{K}}_2^{(2)}(p) = \widetilde{C} \star \widetilde{v} \star \widetilde{v}(p) = 2(4\pi)^{-D} \int_1^\infty \mathrm{d}\alpha_1 \int_1^\infty \mathrm{d}\alpha_2 (\alpha_1\alpha_2 + \alpha_1 + \alpha_2)^{-D/2} \exp\left(-\alpha_1\alpha_2 p^2/(\alpha_1\alpha_2 + \alpha_1 + \alpha_2)\right)$. Notice that it is regular function of the variable p^2 in four dimensions.

Since it is marginal, it does not determine the first u-derivative at zero momentum. Furthermore, it requires

$$B'(0) = 0. \tag{117}$$

Eq. (117) can be checked to be false in four dimensions. The problem is a wave function term which is generated dynamically to second order. As it stands the theory is inconsistent. This problem is solved by introduction of a wave function term to first order. Thus we extend (94) to

$$\mathcal{V}^{(1)}(\phi) = \frac{1}{4!}\int \mathrm{d}^D x \phi(x)^4 + \frac{\zeta^{(1)}}{2}\int \mathrm{d}^D x \partial_\mu \phi(x) \partial_\mu \phi(x). \tag{118}$$

A first order wave function term is consistent with (97) in four dimensions. Both terms in (118) are marginal. The first order scaling equation leaves the new parameter $\zeta^{(1)}$ undetermined. We use this freedom to satisfy the second order constraint (117). It is a general feature of this approach that marginal parameters are determined one order later than the other ones to a given order. Nevertheless we have a recursive perturbation theory. It is now consistent to all orders. Having accompanied the ϕ^4-vertex with a first order wave function term, we find a few more effective interactions in the analogue of (100). With (94) replaced by (118), it becomes

$$\begin{aligned} K^{(2)}(\phi) = \frac{\beta^{(2)}}{4!}\int \mathrm{d}^D x \phi(x)^4 + \int \mathrm{d}^D x \mathrm{d}^D y \Big\{ \frac{1}{2}\phi(x)\phi(y) C(x-y) v(x-y)^2 + \\ \frac{1}{4!}\phi(x)^2\phi(y)^2 6C(x-y)v(x-y) + \frac{1}{6!}\phi(x)^3\phi(y)^3 20 C(x-y) \Big\} + \\ \frac{\beta^{(2)}\zeta^{(1)}}{2}\int \mathrm{d}^D x \partial_\mu\phi(x)\partial_\mu\phi(x) + \int \mathrm{d}^D x \mathrm{d}^D y \Big\{ \frac{(\zeta^{(1)})^2}{2}\phi(x)\phi(y) 2(-\triangle_x)(-\triangle_y) C(x-y) + \\ \frac{\zeta^{(1)}}{2}\phi(x)^2 2(-\triangle_y) C(y) v(y) + \frac{\zeta^{(1)}}{4!}\phi(x)\phi(y)^3 8(-\triangle_y) C(y). \end{aligned} \tag{119}$$

It follows that we have to replace eq. (105) by

$$\widetilde{\mathcal{K}}_2^{(2)}(p) = \widetilde{C}\star\widetilde{v}\star\widetilde{v}(p) + \zeta^{(1)} a + \beta^{(2)}\zeta^{(1)} p^2 + 2(\zeta^{(1)})^2 (p^2)^2 \widetilde{C}(p), \tag{120}$$

$$\widetilde{\mathcal{K}}_4^{(2)}(p_1,p_2,p_3) = \beta^{(2)} + 6\widetilde{C}\star\widetilde{v}(p_1+p_2) + 8\zeta^{(1)}\, p_1^2 \widetilde{C}(p_1), \tag{121}$$

$$\widetilde{\mathcal{K}}_6^{(2)}(p_1,\ldots,p_5) = 20\widetilde{C}(p_1+p_2+p_3). \tag{122}$$

The constant a is here given by a the convergent one loop integral

$$a = 2\int \mathrm{d}^D y (-\triangle_y) C(y) v(y) = 4\int \frac{\mathrm{d}^D p}{(2\pi)^D}\exp(-2p^2) = (4\pi)^{-2}, \quad D = 4. \tag{123}$$

We then proceed exactly as above. The new terms in (122) do not disturb the recursion. The six point kernel is not affected at all. Neither is the four point kernel at zero momentum. Therefore the coefficent $\beta^{(2)}$ is independent of the new constant. The quartic remainder gets an extra contribution and requires first knowledge of $\zeta^{(1)}$. Spelled out explicitely, the eq. (117) becomes

$$0 = \beta^{(2)}\zeta^{(1)} + \frac{\partial}{\partial(p^2)}\widetilde{C} \star \widetilde{v} \star \widetilde{v}(p)\Big|_{p=0}. \tag{124}$$

It determines the first order wave function parameter. The second order work is therefore best organized as follows. We first compute $\beta^{(2)}$ from (107), and second $\zeta^{(1)}$ from (124). Knowing of the marginal data we then compute all kernels in terms of their Taylor expansions and remainders. In the normal ordered formulation the order in which the kernels are computed is of no importance. The value of the first order wave function paramter comes out as

$$\zeta^{(1)} = \frac{-1}{18(4\pi)^2}. \tag{125}$$

The second order effective mass parameter is then changed from (113) to

$$A(0) = \frac{1}{(4\pi)^4}\left(2\log(2) - \log(3) - \frac{1}{36}\right). \tag{126}$$

The changed remainder terms of the quadratic and the quartic kernel follow immediately from (122). This completes the calculation of the second order. In summary we have extracted the non-irrelevant part by Taylor expansion. The Taylor coefficients have been determined directly by evaluation at zero momentum. The irrelevant part, including the Taylor remainders, have been obtained by integration of the corresponding scaling equations.

4.2.4 Higher Orders

The second order scheme generalizes to third and higher orders. Assume that we have computed $\mathcal{V}^{(s)}(\phi)$ and $\beta^{(s)}$ to all orders $1 \leq s \leq r-1$ except for $\zeta^{(r-1)}$. Then we first compute $\beta^{(r)}$, thereafter $\zeta^{(r-1)}$, and then the $\mathcal{V}^{(r)}(\phi)$ except for $\zeta^{(r)}$. The non-irrelevant degrees of freedom are again separated out by Taylor expansion in momentum space. A convenient notation for this Taylor expansion goes as follows. Introduce scaling fields

$$\mathcal{O}_{2,0}(\phi) = \frac{1}{2}\int \mathrm{d}^D x\ \phi(x)^2, \quad \mathcal{O}_{2,2}(\phi) = \frac{1}{2}\int \mathrm{d}^D x\ \partial_\mu\phi(x)\partial_\mu\phi(x), \tag{127}$$

$$\mathcal{O}_{4,0}(\phi) = \frac{1}{4!}\int \mathrm{d}^D x\ \phi(x)^4, \tag{128}$$

with scaling dimensions

$$\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) \mathcal{O}_{2,0}(\phi) = 2\mathcal{O}_{2,0}(\phi), \quad \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) \mathcal{O}_{2,2}(\phi) = 0, \tag{129}$$

$$\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) \mathcal{O}_{4,0}(\phi) = (4-D)\mathcal{O}_{2,0}(\phi). \tag{130}$$

In four dimensions, the non-irrelevant part of the effective potential can be written as a sum

$$\mathcal{V}^{(r)}_{rel}(\phi) = \mu^{(r)}\mathcal{O}_{2,0}(\phi) + \zeta^{(r)}\mathcal{O}_{2,2}(\phi) + \lambda^{(r)}\mathcal{O}_{4,0}(\phi). \tag{131}$$

The general form of the effective potential to order r in the minimal scheme is

$$\mathcal{V}^{(r)}(\phi) = \sum_{n=1}^{r+1} \frac{1}{(2n)!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_{2n}\ \phi(x_1)\ldots\phi(x_{2n})\ \mathcal{V}^{(r)}_{2n}(x_1, \ldots, x_{2n}). \tag{132}$$

The non-irrelevant coupling constants are, in terms of these kernels, given by

$$\mu^{(r)} = \int \mathrm{d}^D x_2\ \mathcal{V}^{(r)}_2(x_1, x_2) = \widetilde{\mathcal{V}}^{(r)}_2(0), \tag{133}$$

$$\zeta^{(r)} = \frac{-1}{2D} \int \mathrm{d}^D x_2\ (x_1 - x_2)^2\ \mathcal{V}^{(r)}(x_1, x_2) = \frac{\partial}{\partial(p^2)} \widetilde{\mathcal{V}}^{(r)}_2(p)\bigg|_{p=0}, \tag{134}$$

$$\lambda^{(r)} = \int \mathrm{d}^D x_2\ \mathrm{d}^D x_3\ \mathrm{d}^D x_4\ \mathcal{V}^{(r)}_4(x_1, x_2, x_3, x_4) = \widetilde{\mathcal{V}}^{(r)}_4(0,0,0). \tag{135}$$

A compact notation for their extraction from the polynomial (132) is the formal pairing

$$\mu^{(r)} = \left(\mathcal{O}_{2,0}, \mathcal{V}^{(r)}\right), \quad \zeta^{(r)} = \left(\mathcal{O}_{2,2}, \mathcal{V}^{(r)}\right), \quad \lambda^{(r)} = \left(\mathcal{O}_{4,0}, \mathcal{V}^{(r)}\right). \tag{136}$$

Its meaning is, as was said before, nothing but Taylor expansion in momentum space. The scaling equation to order r can be put into the form

$$\left\{ r\beta^{(1)} - \left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) \right\} \mathcal{V}^{(r)}(\phi) = -\mathcal{K}^{(r)}(\phi). \tag{137}$$

The right hand side of (137) still depends on $\beta^{(r)}$ and $\zeta^{(r-1)}$. It is given by

$$\mathcal{K}^{(r)}(\phi) = \sum_{s=2}^{r} \binom{r}{s} \beta^{(s)} \mathcal{V}^{(r-s+1)}(\phi) + \sum_{s=1}^{r-1} \binom{r}{s} \left\langle \mathcal{V}^{(s)}, \mathcal{V}^{(r-s)} \right\rangle. \tag{138}$$

Projecting both sides of (137) to $\mathcal{O}_{4,0}(\phi)$ we find

$$(r-1)\beta^{(1)}\lambda^{(r)} = -\beta^{(r)} - \sum_{s=2}^{r-1} \binom{r}{s} \beta^{(s)}\lambda^{(r-s+1)} - \sum_{s=1}^{r-1} \binom{r}{s} \left(\mathcal{O}_{4,0}, \left\langle \mathcal{V}^{(s)}, \mathcal{V}^{(r-s)} \right\rangle\right). \tag{139}$$

Since $\lambda^{(r)} = 0$ by construction, eq. (139) determines the coefficient $\beta^{(r)}$. The right hand side of (139) does not depend on $\zeta^{(r-1)}$ since $\left(\mathcal{O}_{4,0}, \left\langle \mathcal{V}^{(1)}(\phi), \mathcal{O}_{2,2} \right\rangle\right) = 0$. Thus the order $r-1$ wave function term does not contribute through the renormalization group bracket to an effective ϕ^4-vertex. Thereafter projecting both sides of (137) to $\mathcal{O}_{2,2}(\phi)$ we find

$$r\beta^{(1)}\zeta^{(r)} = -\binom{r}{2} \beta^{(2)}\zeta^{(r-1)} - \sum_{s=3}^{r} \binom{r}{s} \beta^{(s)}\zeta^{(r-s+1)} - \sum_{s=1}^{r-1} \binom{r}{s} \left(\mathcal{O}_{2,2}, \left\langle \mathcal{V}^{(s)}, \mathcal{V}^{(r-s)} \right\rangle\right). \tag{140}$$

In four dimensions the left hand side is zero since $\beta^{(1)} = 4 - D = 0$. But then (140) determines the value of $\zeta^{(r-1)}$. Notice that $\beta^{(2)}$ is not zero. Notice further that the renormalization group bracket does not depend on $\zeta^{(r-1)}$ because $\left(\mathcal{O}_{2,2}, \left\langle \mathcal{V}^{(1)}, \mathcal{O}_{2,2} \right\rangle\right) = 0$. The reason is that the covariance C is regular at zero momentum. The rest of the work is immediately put to order. The effective mass parameter to order r follows from

$$\left(r\beta^{(1)} - 2\right)\mu^{(r)} = -\sum_{s=2}^{r} \binom{r}{s} \beta^{(s)}\mu^{(r-s+1)} - \sum_{s=1}^{r-1} \binom{r}{s} \left(\mathcal{O}_{2,0}, \left\langle \mathcal{V}^{(s)}, \mathcal{V}^{(r-s)} \right\rangle\right). \tag{141}$$

Its computation requires the extraction of the effective mass term in the renormalization group bracket. The computation of $\mathcal{V}^{(r)}_{rel}(\phi)$ except for $\zeta^{(r)}$ is complete. The irrelevant part is directly integrated following exactly the scheme of the second order calculation. Eq. (138) can be expanded into a polynomial

$$\mathcal{K}^{(r)}(\phi) = \sum_{n=1}^{r+1} \frac{1}{(2n)!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_{2n}\, \phi(x_1) \ldots \phi(x_{2n})\, \mathcal{K}^{(2r)}(x_1, \ldots, x_{2n}). \tag{142}$$

The scaling equations for the Fourier transformed kernels are given by

$$\left\{ -\widetilde{\sigma}^{(r)}_{2n} + \sum_{m=1}^{2n-1} p_m \frac{\partial}{\partial p_m} \right\} \widetilde{\mathcal{V}}^{(r)}_{2n}(p_1, \ldots, p_{2n-1}) = -\widetilde{\mathcal{K}}^{(r)}_{2n}(p_1, \ldots, p_{2n-1}). \tag{143}$$

For $n \geq 3$, the scaling dimension is $\widetilde{\sigma}^{(r)}_{2n} = D + n(2-D) - r(4-D) = 4 - 2n < 0$, $D = 4$. Those kernels are therefore all irrelevant. The scaling equation is in this case integrated to

$$\widetilde{\mathcal{V}}^{(r)}_{2n}(p_1, \ldots, p_{2n-1}) = -\int_0^1 \frac{\mathrm{d}t}{t} t^{-\widetilde{\sigma}^{(r)}_{2n}}\, \widetilde{\mathcal{K}}^{(r)}_{2n}(tp_1, \ldots, tp_{2n-1}). \tag{144}$$

The integral converges because of the negative power counting. The zero momentum part of the four point kernel has already successfully been transfered to the β-function. Its remainder is irrelevant and integrated to

$$\widetilde{\mathcal{V}}_4^{(r)}(p_1, p_2, p_3) = -\int_0^1 \frac{\mathrm{d}t}{t} \left\{ \widetilde{\mathcal{K}}_4^{(r)}(tp_1, tp_2, tp_3) - \widetilde{\mathcal{K}}_4^{(r)}(0,0,0) \right\}. \tag{145}$$

The integral converges due to the subtraction at zero momentum. Finally, the two point kernel is reconstructed with the help of

$$\widetilde{\mathcal{V}}_2^{(r)}(p) = A(p^2), \quad \widetilde{\mathcal{K}}_2^{(r)}(p) = 2B(p^2), \tag{146}$$

and

$$A(u) = \mu^{(r)} + \zeta^{(r)} u + \frac{u^2}{2} \int_0^1 \mathrm{d}s (1-s) A''(su) \tag{147}$$

through

$$A''(u) = -\int_0^1 \mathrm{d}t B''(tu). \tag{148}$$

The scheme is now complete. We have a manifestly finite recursive local perturbation theory.

5 Regularity

The integration of renormalization group differential equations generally requires initial data. In the local expansion, where we no more have evolution equations, we substitute the initial data by a requirement that the solutions be finite and regular.

5.1 Renormalization Group PDEs

In the perturbation expansion the recursion to each higher order consists of solving a set of renormalization group PDEs of the general form

$$\left\{ p \frac{\partial}{\partial p} - \sigma \right\} F(p) = G(p). \tag{149}$$

Here $\sigma \in \mathbb{Z}$, and $G(p)$ is a given function of $p \in \mathbb{R}^N$. In this section we will have a look at the *regular* solutions of (149).

5.1.1 Irrelevant Case

The irrelevant case is defined by $\sigma < 0$. Let us assume that $G(p)$ is a continuous function on $\mathbb{R}^N$. Eq. (149) is equivalent to

$$L\frac{\mathrm{d}}{\mathrm{d}L}\left\{L^{-\sigma}F(Lp)\right\} = L^{-\sigma}G(Lp). \tag{150}$$

A special solution to (149) is then given by

$$F(p) = \int_0^1 \frac{\mathrm{d}L}{L}L^{-\sigma}G(Lp). \tag{151}$$

Let us require the solution to be a continuous differentiable function on $\mathbb{R}^N$. Then we notice the following facts:

I) There exists a unique solution to (149). II) It is given by (151).

First, suppose that we have two different solutions $F_1(p)$ and $F_2(p)$ of (149) which are both continuous differentiable. Their difference satisfies

$$\left\{p\frac{\partial}{\partial p} - \sigma\right\}(F_1(p) - F_2(p)) = 0. \tag{152}$$

The only solution to this equation, which is regular at the origin in the case $\sigma < 0$, is zero. Second, (151) is continuous differentiable and is a solution to (149).

Thus we can substitute initial or boundary data by regularity to obtain a unique solution. Notice that its value at zero is $-\sigma F(0) = G(0)$, as can be seen from both (149) and (151).

5.1.2 Relevant Case

The relevant case is defined by $\sigma > 0$. In this case we cannot use (151) because the integral diverges, unless $G(p)$ provides a sufficiently high power of p. The trick is to perform a Taylor expansion with remainder term to high enough order. Let us assume that $G(p)$ is $(\sigma + 1)$-times continuous differentiable.

Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ be an integer valued multi-index. Define $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \ldots \alpha_N!$, and $p^\alpha = p_1^{\alpha_1} \ldots p_N^{\alpha_N}$. Then $F(p)$ is a solution to (149) iff its derivatives with $\sigma \leq |\alpha|$ satisfy

$$\left\{p\frac{\partial}{\partial p} - (\sigma - |\alpha|)\right\}\frac{\partial^{|\alpha|}F}{\partial p^\alpha}(p) = \frac{\partial^{|\alpha|}G}{\partial p^\alpha}(p). \tag{153}$$

Thus each momentum derivative reduces the power counting parameter by one unit. Solutions to (153) which are regular at the origin satisfy

$$-(\sigma - |\alpha|)\frac{\partial^{|\alpha|}F}{\partial p^\alpha}(0) = \frac{\partial^{|\alpha|}G}{\partial p^\alpha}(0). \tag{154}$$

Their Taylor coefficients are all determined except for those with $\sigma - |\alpha| = 0$, the marginal ones. Let us assume that

$$\frac{\partial^{|\alpha|} G}{\partial p^{\alpha}}(0) = 0, \quad \sigma - |\alpha| = 0. \tag{155}$$

Then (149) can be solved in terms of a Taylor expansion of order σ with remainder,

$$F(p) = \sum_{|\alpha| \leq \sigma} \frac{p^{\alpha}}{\alpha!} \frac{\partial^{|\alpha|} F}{\partial p^{\alpha}}(0) + \sum_{|\alpha| = \sigma + 1} \frac{p^{\alpha}}{\alpha!} \int_0^1 \mathrm{d}t (1-t)^{\sigma} \frac{\partial^{|\alpha|} F}{\partial p^{\alpha}}(tp). \tag{156}$$

The derivatives of order $\sigma + 1$ have negative power-counting and are integrated as above. Let us require the solution to be $(\sigma + 1)$-times continuous differentiable. Then we have:

I) There exists a set of solutions to (149) which can be parametrized by its Taylor coefficients with $\sigma = |\alpha|$. II) The relevant Taylor coefficients with $\sigma < |\alpha|$ are uniquely determined by (154). III) The Taylor remainder is reconstructed from the $(\sigma + 1)$th derivatives. It is unique and follows from (151).

5.2 Large Momentum Bound

We prove a large momentum bound for the solution to the renormalization group PDE under the assumption of a large momentum bound on the inhomogeneous side. We choose an $L_{\infty,\epsilon}$-norm for some $\epsilon > 0$. It is rather wasteful but suffices to prove finiteness of the bilinear renormalization group bracket.

5.2.1 Irrelevant Case

Let $\sigma < 0$. Suppose that the function $G(p)$ in (149) has a finite $L_{\infty,\epsilon}$-norm

$$\|G\|_{\infty,\epsilon} = \sup_{p \in \mathbb{R}^N} \left\{ |G(p)| e^{-\epsilon |p|} \right\} < \infty. \tag{157}$$

Then the solution (151) inherits an $L_{\infty,\epsilon}$-bound. From

$$|F(p)| e^{-\epsilon|p|} \leq \int_0^1 \frac{\mathrm{d}L}{L} L^{-\sigma} |G(Lp)| e^{-\epsilon|p|} \leq \int_0^1 \frac{\mathrm{d}L}{L} L^{-\sigma} e^{-(1-L)\epsilon|p|} \|G\|_{\infty,\epsilon} \tag{158}$$

it follows that

$$\|F\|_{\infty,\epsilon} \leq \frac{1}{-\sigma} \|G\|_{\infty,\epsilon}. \tag{159}$$

Eq. (159) shows that the irrelevant solution to the renormalization group PDE is not only finite but also decreases in the $L_{\infty,\epsilon}$-norm.

5.2.2 Marginal Case

Let $\sigma = 0$. In this case we assemble $F(p)$ using a first order Taylor formula. Suppose then that we have $L_{\infty,\epsilon}$-bounds on the first derivatives

$$\|G_\mu\|_{\infty,\epsilon} = \sup_{p\in\mathbb{R}^N} \left\{ \left| \frac{\partial}{\partial p^\mu} G(p) \right| e^{-\epsilon|p|} \right\} < \infty. \tag{160}$$

If $F(p)$ is marginal, then its first derivatives are irrelevant with scaling dimension minus one. It follows that

$$\|F_\mu\|_{\infty,\epsilon} \leq \|G_\mu\|_{\infty,\epsilon}. \tag{161}$$

Therefrom it follows that

$$\begin{aligned} |F(p)|e^{-\epsilon|p|} &\leq |F(0)|e^{-\epsilon|p|} + \sum_\mu |p_\mu| \int_0^1 \mathrm{d}t |F_\mu(tp)| e^{-\epsilon|p|} \\ &\leq |F(0)| + \sum_\mu |p_\mu| \int_0^1 \mathrm{d}t e^{-(1-t)\epsilon|p|} \|F_\mu\|_{\infty,\epsilon}. \end{aligned} \tag{162}$$

The result is an $L_{\infty,\epsilon}$-bound

$$\|F\|_{\infty,\epsilon} \leq |F(0)| + \frac{1}{\epsilon} \sum_\mu \|F_\mu\|_{\infty,\epsilon}. \tag{163}$$

This estimate is not uniform in ϵ. It works for ϵ arbitrary small, but the bound grows with an inverse power of ϵ. The large momentum growth is a consequence of the split in derivatives and Taylor remainder.

5.2.3 Relevant Case

Let $\sigma > 0$. This case requires a generalization of the bound in the marginal case. The Taylor expansion is pushed to order $\sigma + 1$. Then the derivatives become irrelevant. We assume $L_{\infty,\epsilon}$-estimates on all derivatives

$$\|G_\alpha\|_{\infty,\epsilon} = \sup_{p\in\mathbb{R}^N} \left\{ \left| \frac{\partial^{|\alpha|}}{\partial p^\alpha} G(p) \right| e^{-\epsilon|p|} \right\} < \infty \tag{164}$$

of order $|\alpha| = \sigma + 1$. Since they are irrelevant with scaling dimension minus one it follows that the corresponding derivatives of $F(p)$ obey

$$\|F_\alpha\|_{\infty,\epsilon} \leq \|G_\alpha\|_{\infty,\epsilon}, \tag{165}$$

and are also $L_{\infty,\epsilon}$-bounded. From the Taylor formula it then follows that

$$
\begin{aligned}
|F(p)|e^{-\epsilon|p|} &\leq \sum_{|\alpha|\leq\sigma} \frac{|p^\alpha|}{\alpha!} e^{-\epsilon|p|}|F_\alpha(0)| + \sum_{|\alpha|=\sigma+1} \frac{|p^\alpha|}{\alpha!} \int_0^1 \mathrm{d}t(1-t)^\sigma |F_\alpha(tp)|e^{-\epsilon|p|} \\
&\leq \sum_{|\alpha|\leq\sigma} \frac{|p|^{|\alpha|}}{\alpha!}|F_\alpha(0)| + \sum_{|\alpha|=\sigma+1} \frac{|p|^{\sigma+1}}{\alpha!} \int_0^1 \mathrm{d}t(1-t)^\sigma e^{-(1-t)\epsilon|p|}\|F_\alpha\|_{\infty,\epsilon},
\end{aligned}
\tag{166}
$$

and thus

$$
\|F\|_{\infty,\epsilon} \leq \sum_{|\alpha|\leq\sigma} \frac{1}{\alpha!} A_{\epsilon,|\alpha|}|F_\alpha(0)| + \sum_{|\alpha|=\sigma+1} \frac{1}{\alpha!} B_{\epsilon,\sigma+1}\|F_\alpha\|_{\infty,\epsilon}, \tag{167}
$$

with constants

$$
A_{\epsilon,|\alpha|} = \sup_{p\in\mathbb{R}^N} \left\{|p|^{|\alpha|}e^{-\epsilon|p|}\right\}, \quad B_{\epsilon,\sigma+1} = \frac{\Gamma(\sigma+1)}{\epsilon^{\sigma+1}}. \tag{168}
$$

Thus we again have an $L_{\infty,\epsilon}$-bound on the function $F(p)$. This completes the large momentum bound on $F(p)$. Exactly the same strategy applies to the derivatives of $F(p)$ as well. The irrelevant derivatives inherit immediately large momentum bounds. The relevant derivatives require Taylor expansions. We omit to spell out explicitely the necessary bounds on the derivatives of $G(p)$.

5.3 Iteration and Regularity

The iterative scheme determines order by order $\beta^{(s)}$, $\zeta^{(s-1)}$, $\mu^{(s)}$, and the irrelevant remainders $\widetilde{\mathcal{V}}^{(s)}_{irr,2n}(p_1,\ldots,p_{2n-1})$. It is finite to all orders of perturbation theory because of the following iteration of regularity. Suppose that we have shown the following to all orders $s \leq r-1$:

I) $\beta^{(s)}$, $\zeta^{(s-1)}$, *and* $\mu^{(s)}$ *are finite numbers. II)* $\widetilde{\mathcal{V}}^{(s)}_{irr,2n}(p_1,\ldots,p_{2n-1})$ *is a smooth function on* $\mathbb{R}\times\cdots\times\mathbb{R}$ *for all* $1\leq n\leq s+1$, *symmetric in the momenta, and* $O(D)$*-invariant. III)* $\|\widetilde{\mathcal{V}}^{(s)}_{irr,2n,\alpha}\|_{\infty,\epsilon}$ *is finite for all* $\epsilon>0$, $1\leq n\leq s+1$, *and* $|\alpha|\geq 0$. *Here* α *is a multi-index which labels momentum derivatives.*

Then the same statements hold at order $s=r$. Since they are trivially fulfilled to order one they iterate to all orders of perturbation theory.

To prove the iteration of regularity we once more inspect each step of the iterative scheme. First, the irrelevant remainders $\widetilde{\mathcal{K}}^{(r)}_{irr,2n}(p_1,\ldots,p_{2n-1})$ are smooth functions on $\mathbb{R}\times\cdots\times\mathbb{R}$, symmetric under permutations and $O(D)$-invariant. They and all their momentum derivatives satisfy $L_{\infty,\epsilon}$-bounds. They are composed of two contributions. The first immediately inherits a bound from the induction hypothesis.

The second is a sum of renormalization group brackets of lower orders. Therefore, they consist of multiple convolutions with propagators. The integrals converge, are smooth functions of the external momenta, and satisfy $L_{\infty,\epsilon}$-bounds. Second, we have linear equations for the coefficients $\beta^{(r)}$, $\zeta^{(r-1)}$, and $\mu^{(r)}$ with finite coefficients. Third, the integration of the inhomogeneous renormalization group PDEs, yields solutions with the desired properties.

6 Conclusions

The aim of perturbative renormalization theory is to derive power series expansions for Green's functions which are free of divergencies. The BPHZ theorem states that this can be accomplished by writing the Green's functions in terms of renormalized parameters. An elegant proof of the BPHZ theorem was given by Callan [C76]. A polished version of which is due to Lesniewski [L83]. Their method is similar to ours in that it is based on renormalization group equations for the renormalized Green's functions, the Callan-Symanzik equations. In some sense (1) is a Wilson-analogue of the Callan-Symanzik equations. The method proposed here is different in that it does *not* resort to any kind of graphical analysis, not to analysis of sub-graphs, and not to skeleton expansions.

A new generation of proofs of the BPHZ theorem was initiated with the work of Polchinski [P84]. His proof has been simplified further by Keller, Kopper, and Salmhofer [KKS90]. Their approach is similar to the method advocated here in that it is based on Wilson's exact renormalization differential equation. The details are however quite different. The main difference is that Polchinski begins with a cutoff theory. He then shows how the cutoff can be removed in a way such that the effective interaction remains finite. Our method directly addresses the limit theory without cutoffs, *expressed* in terms of a renormalization group transformation with cutoffs. In some sense we are here simultaneously changing Polchinski's renormalization conditions and integrating an amount of fluctuations. Unlike Polchinski and followers we use a renormalization group differential equation with dilatation term. A way to think of (1) is as a renormalization group fixed point of a system which has been enhanced by one degree of freedom, the running coupling. This fixed point problem can only be formulated with rescaling and with dilatation term.

Another renormalization group approach to renormalized perturbation theory comes from Gallavotti [G85, GN85] and collaborators. Pedagocial accounts of tree expansions can be found in [BG95, FHRW88]. There the result of renormalization is expressed in terms of a renormalized tree expansion. The program of [Wi96] with an iterated transformation with fixed L is related to the tree expansion. Both are built upon a cumulant expansion for the effective interaction. The renormalization

procedure is however quite different. Like Polchinski, Gallavotti starts from a cut-off theory. It is organized in terms of trees, which describe the sub-structure of divergencies in Feynman diagrams. The divergencies are transformed into a flow of the non-irrelevant couplings. This part is similar to ours. The basic difference with Gallavotti is that we do not organize our expansion in terms of trees. A hybrid approach between Polchinski and Gallavotti is due to Hurd [H89].

An important question is wether this construction of renormalized trajectories extends beyond perturbation theory.[6] Another important question is wether it extends to renormalized trajectories at non-trivial fixed points. We hope to return with answers to these questions in the future.

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